Gravitation as a Plastic Distortion of the Lorentz Vacuum

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November 2, 2009

Abstract

In this paper we present a theory of the gravitational field where this field (a kind of 'square root' of $\boldsymbol{g}$) is represented by a $(1,1)$-extensor field $\boldsymbol{h}$ describing a plastic distortion of the Lorentz vacuum (a complex substance that lives in a Minkowski spacetime) due to the presence of matter. The field $\boldsymbol{h}$ distorts the Minkowski metric extensor η generating what may be interpreted as an effective Lorentzian metric extensor $\boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}$ and also it permits the introduction of different kinds of parallelism rules on the world manifold, which may be interpreted as distortions of the parallelism structure of Minkowski spacetime and which may have non null curvature and/or torsion and/or nonmetricity tensors. We thus have different possible effective geometries which may be associated to the gravitational field and thus its description by a Lorentzian geometry is only a possibility, not an imposition from Nature. Moreover, we developed with enough details the theory of multiform functions and multiform functionals that permitted us to successfully write a Lagrangian for $\boldsymbol{h}$ and to obtain its equations of motion, that results in our theory equivalent to Einstein field equations of General Relativity (for all those solutions where the manifold M is diffeomorphic to $\mathbb{R}^4$). However, in our theory, differently from the case of General Relativity, trustful energy-momentum and angular momentum conservation laws exist. We express also the results of our theory in terms of the gravitational potentials $\mathfrak{g}^{\mu} = \boldsymbol{h}^{\dagger}(\vartheta^{\mu})$ where $\{\vartheta^{\mu}\}$ is an orthonormal basis of Minkowski spacetime in order to have results which may be easily expressed with the theory of differential forms. The Hamiltonian formalism for our theory (formulated in terms of the potentials $\mathfrak{g}^{\mu}$) is also discussed. The paper contains also several important Appendices that complete the material in the main text.

Contents

1 Introduction

In this article we present a theory of the gravitational field where this field is described by a distortion field $\boldsymbol{h}$ which lives and interacts with the matter fields in Minkowski spacetime. It describes a plastic deformation of the Lorentz vacuum understood as a real physical substance[1]. The distortion field is represented in the mathematical formalism of this paper—called the multiform and extensor calculus on manifolds ($MECM$)— by a $(1,1)$-extensor field and as we are going to see, $\boldsymbol{h}$ it is a kind of square root of a $(1,1)$-extensor field $\boldsymbol{g}$ which is directly associated to the metric tensor $\mathbf{g}$ which as well known represents important aspects of the gravitational field in Einstein's General Relativity Theory (GRT). Before going into the details we make a digression in order to present the main motivations for our enterprise.

We start by recalling that in GRT the gravitational field has a status which is completely different from the other physical fields. Indeed in GRT the gravitational field is interpreted as aspects of a *geometrical* structure of the spacetime manifold. More precisely we have that each gravitational field generated by a given matter distribution (represented by a given energy-momentum tensor) is represented by a pentuple $(M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow)$, where[2]:

(mi) M is a 4-dimensional Hausdorff manifold which paracompact, connected and noncompact,

(mii) $\mathbf{g} \in \sec T_2^0 M$ is a tensor of signature -2, called a *Lorentzian metric*[3] on M,

(miii) D is the Levi-Civita connection of $\mathbf{g}$,

(miv) M is an oriented by the volume element $\tau_{\mathbf{g}} \in \sec \bigwedge^4 T^*M$ and the symbol $\uparrow$ means that M is also time oriented.

GRT supposes that particles and fields are some special configurations[4] on the manifold M. Particles are described by triples $\langle (m,q), S, \sigma \rangle$ where m $(0 \leq m < \infty)$ is the particle mass, q $(-\infty < q < \infty)$ is a parameter called the electric charge, S is the particle's spin[5], and σ is a regular curve called the world line of the particle.

If $m > 0$ the particle is called a *bradyon* and in this case σ is a *timelike* curve pointing into the future.

If $m = 0$ the particle is said to be a luxon and in this case σ is a *lightlike* curve.

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[1] We recall that deformations of a medium may be of the elastic or plastic type [108]. Later we explain the difference between those two types of deformation.

[2] In fact a gravitational field is defined by an equivalence class of pentuples, where $(M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow)$ and $(M', \mathbf{g}', D', \tau'_{\mathbf{g}}, \uparrow')$ are said equivalent if there is a diffeomorphism $\mathtt{h}: M \to M'$, such that $\mathbf{g}' = \mathtt{h}^*\mathbf{g}$, $D' = \mathtt{h}^* D$, $\tau'_{\mathbf{g}} = \mathtt{h}^*\tau_{\mathbf{g}}, \uparrow' = \mathtt{h}^* \uparrow$, (where $\mathtt{h}^*$ here denotes the pullback mapping). For more details, see, e.g., [86, 79, 82].

[3] The field $\mathbf{g}$ in GRT obeys Einstein's field equation, to be recalled below.

[4] Particles and fields are to be interpreted in future developments of our theory respectively as different kinds of defects and distortions of the Lorentz vacuum.

[5] For details of how to characterize the particle's spin in the Clifford bundle formalism see e.g., [81].

The different physical fields are modelled by special sections of the tensor and spinor bundles over the basic manifold M.

The world lines that represent the history of particles as well as the tensor and spinor fields which describe the physical fields are mathematically described by some system of differential equations (in general called the *equations of motion*) in which the metric tensor $\boldsymbol{g}$ appears.

The energy-momentum content of a system of particles and fields is described by their respective *energy-momentum tensors* $\mathbf{T}_{\mathrm{p}} \in \sec T_0^2 M$ and $\mathbf{T}_{\mathrm{f}} \in \sec T_0^2 M$. As it is well known [66, 86], those objects appear in the second member of Einstein's field equation, whereas in the first member of that equation appears a tensor called the *Einstein tensor*.

Einstein's equation may be written (after a convenient choice of a local chart on $U \subset M$) as a system of ten non linear partial differential equations which contains a certain combination of the first and second order partial derivatives of the components $g_{\mu\nu}$ of the metric tensor.

Einstein's equation is sometimes described in a pictorially way [66] by saying that "the geometry says how the matter must move and by its turn matter says how the geometry must curve". This pictorial description induces pedestrians on the subject to imagine that the *manifold* M which models spacetime is some kind of *curved* hypersurface (a 4-brane living in a $(4+p)$-dimensional pseudo-euclidean space). It is then necessary in order to appreciate the developments of this paper to deconstruct such an idea by explaining in a rigorous way and illustrating with examples what the concept of curvature used in *GRT* really means, and second to realize that the use of this concept in certain gravitational theories is no more than a coincidence.

We recall that from a mathematical point of view a general differential manifold M by itself possess only a topological structure and a differential structure of the charts of its maximal atlas. It must be clear to start, that the topological structure of M is not fixed in *GRT* and indeed the topology of M is introduced 'by hand' in specific problems and situations [22, 86]. On the other hand any given M may support in general many distinct *geometrical structures*.

In what follows we say that a pair (M, ∇) where M is a general differential manifold and ∇ is an arbitrary connection on M is a space endowed with a parallelism rule. We say that the triple $(M, \boldsymbol{g}, \nabla)$ is a *geometric space structure* (*GSS*) and that $\boldsymbol{g}$ is a general metric tensor on M. The signature of a $\boldsymbol{g}$ is arbitrary. So, even in the case where $(M, \boldsymbol{g}, \nabla)$ is part of a Lorentzian spacetime structure and $\boldsymbol{g}$ has signature -2 we can have another *GSS* $(M, \boldsymbol{g}', \nabla)$ where $\boldsymbol{g}'$ has an another signature[6].

Now the objects that characterize a *GSS* are the following tensor fields:[7]:

(egi) *nonmetricity of* ∇, $\mathfrak{A} \in T_3^0 M$,

$$\mathfrak{A} = \nabla \boldsymbol{g}. \tag{1}$$

(egii) *torsion of* ∇, represented by a tensor field $\Theta \in T_2^1 M$,

[6] In particular in our theory we will see that we shall need to introduce a $\mathbf{g}'$ with signature $+4$.

[7] The definitions of those objects will be recalled in Chapter 3.

(egiii) the *Riemannian curvature* of ∇, represented by a tensor field $\mathfrak{R} \in T_3^1 M$.

A geometric space structures is said to be a:

(a) Riemann-Cartan- Weyl *GSS* iff $\mathfrak{A} \neq 0, \Theta \neq 0, \mathfrak{R} \neq 0$,

(b) Riemann-Cartan *GSS* iff $\mathfrak{A} = 0, \Theta \neq 0, \mathfrak{R} \neq 0$,

(c) Riemann *GSS* iff $\mathfrak{A} = \Theta = 0, \mathfrak{R} \neq 0$.

A general *GSS* $(M, \mathbf{g}, \nabla)$ such that $\nabla \mathbf{g} = 0$ will be called a metric compatible geometric space structure (*MCGSS*)

1.1 Flat Spaces and Affine Spaces

A *MCGSS* $(M, \mathbf{g}, \nabla)$ such that $\mathfrak{A} = \Theta = \mathfrak{R} = 0$ is said to be a *globally flat* Riemann *MCGSS*. A *MCGSS* where the parallelism rule is such that $\Theta = \mathfrak{R} = 0$ and where $M \simeq \mathbb{R}^n$ is isomorphic to a (real) affine space of the same dimension, which we denote by $\mathcal{A}^n$.

As it is well known [99] a (real) affine space has as a fundamental property an operation called the difference between two given points, such that if p, $q \in \mathcal{A}^n$ (or $p, q \in \mathbb{E}^n$) then $(p - q) \in \mathbf{V}$, where $\mathbf{V}$ is a real n-dimensional vector space. So, we may say in a pictorially way that an affine space is a vector space from where the origin has been stolen. Given an arbitrary point $o \in \mathcal{A}^n$ (said to be the *origin*) the object $(p - o) \in \mathbf{V}$ is appropriately called the position vector of p relative to o. This operation permit us to define an *absolute parallelism* rule (denoted $\upuparrows$) in an affine space. Given two arbitrary points o,$o' \in \mathcal{A}^n$ consider the set of all tangent vectors at those two points, i.e., $\{(p - o),\ p \in \mathcal{A}^n\}$ and $\{(p' - o'),\ p' \in \mathcal{A}^n\}$. Then we say that a tangent vector at o, say $(p - o)$ is parallel to a tangent vector at o', say $(p' - o')$ if being $(p - o) = \mathbf{v} \in \mathbf{V}$ and $(p' - o') = \mathbf{v}' \in \mathbf{V}$ we have $\mathbf{v} = \mathbf{v}'$. This parallelism rule permit us to identify the pair $(\mathcal{A}^n, \upuparrows)$ with a parallelism structure $(M \simeq \mathbb{R}^n, \nabla^{\upuparrows})$ where $\nabla^{\upuparrows}$ is a connection, called the Euclidean connection and defined as follows. Let $\{\mathrm{x}^i\}$ $i = 1, 2, ..., n$ be standard Cartesian coordinates for $M \simeq \mathbb{R}^n$. Consider the global coordinate basis $\{e_i\}$ for TM where $e_i := \partial/\partial \mathrm{x}^i$. Then, put

$$\nabla^{\upuparrows}_{e_j} e_i = 0. \tag{2}$$

We define an Euclidean *MCGSS* as a triple $\mathbb{E}^n = (\mathbb{R}^n, \boldsymbol{g}, \nabla^{\upuparrows})$ where $\boldsymbol{g}$ defines the standard Euclidean scalar product in each tangent space T_pM by $\left.\mathbf{g}\right|_p (\left.e_i\right|_p, \left.e_j\right|_p) = \delta_{ij}$. For future reference we say that if the signature[8] of $\boldsymbol{g}$ is $(1, n-1)$, i.e. $\left.\mathbf{g}\right|_p (\left.e_i\right|_p, \left.e_j\right|_p) = \eta_{ij}$ where the matrix (η_{ij}) with entries η_{ij} is the diagonal matrix $\mathrm{diag}(1, -1, ..., -1)$.

Now, from the classification of the *GSS* structures given above it is clear that the Riemannian curvature does not refer to any *intrinsic* property of M, but it refers to a *particular* parallelism structure (i.e., a property of a particular connection ∇) that has been defined on M. To keep this in mind is crucial,

[8]A metric with signature $(n - 1, 1)$ is also called pseudo-euclidean. Note also that mathematical textbooks define the signature of a metric (p, q) as the number $s = p - q$.

because very often we observe notable confusions between the concept of Riemannian curvature and the concept of *bending*[9] of hypersurfaces embedded in a higher dimensional euclidean *MCGSS*, something that may lead to equivocated and even psychedelic interpretations concerning the physical and mathematical contents of *GRT*.

So, before proceeding we recall some examples which we hope will clarify the difference between *bending* and *curvature*.

Consider a 2-dimensional cylindrical surface $\mathcal{C}$ embedded in $\mathbb{E}^3$. Using as metric on $\mathcal{C}$ the pullback of the Euclidean metric of $\mathbb{E}^3$ and defining on $\mathcal{C}$ a connection which is the pullback of the connection $\nabla^{\upharpoonleft\upharpoonright}$ of $\mathbb{E}^3$ (which happens to be the Levi-Civita connection of the induced metric) it is possible to verify without difficulties that the Riemannian curvature of $\mathcal{C}$ is *null*, i.e., the cylinder $\mathcal{C}$ is *flat* according to its Riemannian curvature. However a cylinder $S^1 \times \mathbb{R}$ living in $\mathbb{E}^3$ is very different topologically from any plane $\mathbb{R}^2$ which lives in $\mathbb{E}^3$ and which *may* have zero curvature tensor *if* it is part of a *MCGSS* (say ${}^p\mathbb{E}^2$) where the metric on it is the pullback of the Euclidean metric of $\mathbb{E}^3$ and its connection is the pullback of the connection $\nabla^{\upharpoonleft\upharpoonright}$ of $\mathbb{E}^3$ (which happens to be the Levi-Civita connection of the induced metric). As we already said (see Footnote 9) it is the *shape tensor* the mathematical object that helps to characterize distinct r-dimensional differential manifolds (with identical properties concerning the metric and the connection structures) when they are viewed as hypersurfaces embedded in an Euclidean (or pseudo-euclidean) GSS of appropriate dimension $n > r$, and of course, the shape tensors of $\mathcal{C}$ and $\mathbb{R}^2$ (as part of the *MCGSS* ${}^p\mathbb{E}^2$) are very different [72].

Another very instructive example showing the crucial distinction between curvature and bending is the following [20]:

(A) Consider the torus $\mathcal{T}_1$ as a surface embedded in $\mathbb{E}^3$ and represented, e.g., in the canonical coordinates of $\mathbb{E}^3$ by the equation

$$(x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2). \tag{3}$$

Of course, $\mathcal{T}_1$ is a 2-dimensional manifold. If we consider on $\mathcal{T}_1$ a *MCGSS* structure where the metric on $\mathcal{T}_1$ is the pullback of the Euclidean metric of $\mathbb{E}^3$ and its connection is the pullback of the connection $\nabla^{\upharpoonleft\upharpoonright}$ of $\mathbb{E}^3$ (which results to be the Levi-Civita connection of the induced metric) then the Riemann curvature tensor of that pullback connection on $\mathcal{T}_1$ is *non null*, as it is easy to verify.

(B) Now, the subset $\mathcal{T}_2$ de $\mathbb{E}^4$ given by ,

$$\mathcal{T}_2 = \{x \in \mathbf{E}^4 \ / \ (x^1)^2 + (x^2)^2 = (x^3)^2 + (x^4)^2 = 1\}, \tag{4}$$

is diffeomorphic to the torus $\mathcal{T}_1$ defined in **(A)**. Consider the *MCGSS structure defined on* $\mathcal{T}_2$ where the metric is the pullback of the Euclidean metric of $\mathbb{E}^4$ and its connection is the pullback of the connection $\nabla^{\upharpoonleft\upharpoonright}$ of $\mathbb{E}^4$. It can be easily verified that such *MCGSS* is globally flat, i.e., $\Theta = \mathfrak{R} = 0$.

[9] We recall that the bending of a r-dimensional hypersurface S considered as a subset of points of an euclidean (or pseudo-eucldean) *GSS* of appropriated dimension [12] n is characterized by the so called *shape tensor* [47, 72].

Those two examples leave clear that we should not confound curvature with bending. Indeed, those examples show that a given manifold when interpreted as a r-dimensional hypersurface embedded in an euclidean *MCGSS* of appropriate dimension $n > r$ may be or may be not curved (even if that *MCGSS* heritages as metric tensor and connection the pullback of the metric tensor and the connection of $\mathbb{E}^n$). In particular keep in mind that a 2-dimensional torus in order to be part of a flat *MCGSS* (with a Levi-Civita connection of the pullback metric) must be embedded in a euclidean *GSS* with more dimensions than in the case in which it is part of a non flat *MCGSS* (with a Levi-Civita connection of the pullback metric).[10]

We give yet another example (which Cartan showed to Einstein in 1922 when he visited Paris [17, 39]) to clarify (we hope) the question of the curvature of a given *MCGSS*. Consider the punctured sphere $\mathring{S}^2 = \{S^2-$ north and south poles$\} \subset \mathbb{E}^3$. $\mathring{S}^2$ is clearly a bent surface embedded in $\mathbb{E}^3$. As well known it is a surface with non zero Riemann curvature tensor when interpreted as part of a *MCGSS* $\{\mathring{S}^2, \boldsymbol{g}, D\}$, where the metric $\boldsymbol{g}$ and the connection D are respectively the pullback of the metric and the connection of the *MCGSS* $\mathbb{E}^3$.

Besides the *MCGSS* $\{\mathring{S}^2, \boldsymbol{g}, D\}$ it is possible to define on $\mathring{S}^2$ a Riemann-Cartan *MCGSS* $\{\mathring{S}^2, \boldsymbol{g}, \nabla\}$ where $\boldsymbol{g}$ is as before and ∇ is a metric compatible Riemann-Cartan connection such that its Riemann curvature is null, but its torsion tensor is *non null*. ∇ is called in [67] the navigator connection and the denomination Columbus connection is also sometimes used. In [82] ∇ has been appropriately called the Nunes connection[11]. The parallelism rule defining ∇ is very simple (see Figures 2 and 3). Given a vector $\mathbf{v}$ at p, let α be the angle it makes with the tangent vector to the latitude lines that pass through p. Then $\mathbf{v}$ is said to be parallel transported according to the Nunes connection from p to q along any curve containing those points if at point in the curve the transported vector makes the same angle α with the tangent vector to the latitude line at that point. It is then possible to verify that indeed the Riemann curvature of is null and its torsion is non null. Details are given in Appendix B

[10] We can also put on a torus living in $\mathbb{E}^3$ a non metric compatible connection with null torsion and curvature torsions or yet a metric compatible connection with non null torsion tensor and null curvature torsion. See Appendix A.

[11] Pedro Salacience Nunes (1502–1578) was one of the leading mathematicians and cosmographers of Portugal during the Age of Discoveries. He is well known for his studies in Cosmography, Spherical Geometry, Astronomic Navigation, and Algebra, and particularly known for his discovery of loxodromic curves and the nonius. Loxodromic curves, also called rhumb lines, are spirals that converge to the poles. They are lines that maintain a fixed angle with the meridians. In other words, loxodromic curves directly related to the construction of the Nunes connection. A ship following a fixed compass direction travels along a loxodromic, this being the reason why Nunes connection is also known as navigator connection. Nunes discovered the loxodromic lines and advocated the drawing of maps in which loxodromic spirals would appear as straight lines. This led to the celebrated Mercator projection, constructed along these recommendations. Nunes invented also the Nonius scales which allow a more precise reading of the height of stars on a quadrant. The device was used and perfected at the time by several people, including Tycho Brahe, Jacob Kurtz, Christopher Clavius and further by Pierre Vernier who in 1630 constructed a practical device for navigation. For some centuries, this device was called nonius. During the 19th century, many countries, most notably France, started to call it vernier. More details in http://www.mlahanas.de/Stamps/Data/Mathematician/N.htm.

where a comparison of the transport rules on $\mathring{S}^2$. according to the Levi-Civita and the Nunes connection is also given. From Appendix B it is clear that the orthonormal basis $\{\mathbf{e_1} = \frac{1}{r}\partial/\partial\theta, \mathbf{e_2} = \frac{1}{r\sin\theta}\partial/\partial\phi\}$ (where r, θ,ϕ are the usual spherical coordinates) used in the calculations are not defined on the poles. This is due to the well known fact that S^2 does not admit two linear independent tangent vector fields in all its point. Any given vector field on S^2 necessarily is zero at some point of S^2 (see, e.g., [25]).

A n-dimensional manifold M which admits n linearly independent vector fields $\{e_i \in \sec TM$, $i = 1, 2, ..., n\}$ at all its points is said to be *parallelizable*.

A Riemann-Cartan *MCGSS* $\{M, \mathbf{g}, \nabla\}$ which is also parallelizable and such that there exists a set of n linearly independent vector fields $\{e_i \in \sec TM$, $i = 1, 2, ..., n\}$ on M such that

$$\nabla_{e_j} e_i = 0,\ \forall i, j = 1, 2, ..., n \tag{5}$$

is said to be a *teleparallel MCGSS*. As it may be verified without difficulty any teleparallel *MCGSS* has null Riemannian curvature tensor but non null torsion tensor.

It is also an easy task to invent examples where a manifold M diffeomorphic to $\mathbb{R}^n$ is part of a *MCGSS'* such that its connection has non zero curvature and/or torsion. A simple example is the following. Take an arbitrary global $\mathbf{g}$-orthonormal non coordinate basis for TM ($M \simeq \mathbb{R}^n$) given by $\{\mathbf{f}_i\}$, with $\mathbf{f}_i = f_i^j \partial/\partial \mathrm{x}^j$, and introduce on $M \simeq \mathbb{R}^n$ a connection ∇ such that $\nabla_{\mathbf{f}_j}\mathbf{f}_i = 0$. It is then easy to verify that the *nonmetricity* of that connection, i.e., $\nabla \boldsymbol{g} = 0$, the Riemann curvature tensor of ∇ is null but the torsion tensor of ∇ is non null. This last example is important regarding the *interpretation* of the gravitational field in theories that can be shown to be mathematically equivalent to *GRT* (in a precise sense to be disclosed in due course), *not* as an element describing a particular geometrical property of a given *MCGSS*, but as a *physical field* in the sense of Faraday (i.e., a field with ontology similar to the electromagnetic field) living on Minkowski spacetime. But a reader may ask: is there any *serious* reason for trying such an interpretation for the gravitational field? The answer is yes and will be briefly discussed now.

1.2 Killing Vector Fields, Symmetries and Conservation Laws

We start this section by given some specialized names for structures that may represent a spacetime in *GRT* and some of its more known generalizations. Let then M be a 4-dimensional differentiable manifold satisfying the properties required in the definition given above for a (Lorentzian) spacetime of *GRT*. Consider the structure given by the pentuple $(M, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$, where $\boldsymbol{g}$ is a Lorentzian metric (with signature -2) on M and ∇ is an arbitrary connection on M.

We say that the pentuple $(M, \mathbf{g}, \nabla, \tau_{\boldsymbol{g}}, \uparrow)$ is:

(a) Lorentz-Cartan-Weyl spacetime *iff* $\mathfrak{A} \neq 0, \Theta \neq 0, \mathfrak{R} \neq 0$,

(b) Lorentz-Cartan spacetime *iff* $\mathfrak{A} = 0, \Theta \neq 0, \mathfrak{R} \neq 0$,

(c) Lorentzian spacetime *iff* $\mathfrak{A} = \Theta = 0$, $\mathfrak{R} \neq 0$,
(d) Minkowski spacetime *iff* $M \simeq \mathbb{R}^4$, $\mathfrak{A} = \Theta = \mathfrak{R} = 0$.

In a Minkowski spacetime we will denote the metric tensor by $\boldsymbol{\eta}$. Moreover, we recall that in such spacetime there exists an equivalence class of global coordinate systems for $M \simeq \mathbb{R}^4$ such that if $\{\mathtt{x}^\alpha\}$ is one of these systems then

$$\boldsymbol{\eta} = \eta_{\alpha\beta} d\mathtt{x}^\alpha \otimes \mathtt{x}^\beta, \tag{6}$$

with $\eta_{\alpha\beta} = \boldsymbol{\eta}(\frac{\partial}{\partial \mathtt{x}^\alpha}, \frac{\partial}{\partial \mathtt{x}^\beta})$, where $\alpha, \beta = 0, 1, 2, 3$ and the matrix $(\eta_{\alpha\beta})$ is:

$$(\eta_{\alpha\beta}) = \operatorname{diag}(1, -1, -1, -1). \tag{7}$$

Now, consider a Lorentzian spacetime. Let $\mathbf{T} \in \sec T_s^r M$ be an arbitrary differentiable tensor field. We say that a diffeomorphism $l : U \to U$ $(U \subset M)$ generated by a one-parameter group of diffeomorphisms characterized by the vector field ξ is a *symmetry* of $\mathbf{T}$ iff

$$\pounds_\xi \mathbf{T} = 0, \tag{8}$$

where $\pounds_\xi$ is the Lie derivative [12] in the direction of ξ.

This means that the pullback field $l^* \mathbf{T}$ satisfies

$$l^* \mathbf{T} = \mathbf{T}. \tag{9}$$

For Lorentzian spacetimes (where $\dim M = 4$) the symmetries of the metric tensor $\boldsymbol{g}$ play a very important role. Indeed, it can be shown that the equation. (see, e.g., [59])

$$\pounds_\xi \boldsymbol{g} = 0, \tag{10}$$

called Killing equation can have the maximum number of ten Killing vector fields, and that maximum number ten only occurs for Lorentzian spacetimes with have constant *scalar curvature.* There are only three distinct Lorentzian spacetimes with the maximum number of Killing vector fields: the Minkowski spacetime, the de Sitter spacetime and the anti de Sitter spacetime.

As well known Minkowski spacetime is the mathematical structure used as the 'arena' for the classical theories of fields and particles and also for the so called relativistic quantum field theories [5].

In the classical relativistic theories of fields and particles it can be shown that there exists trustworthy conservation laws for the energy-momentum, angular momentum and conservation of the center of mass for any system of particles and fields[13]. The proof [14] of that statement depends crucially on the existence of the ten Killing vector fields of the Minkowski metric tensor $\boldsymbol{\eta}$.

[12] For details, see, e.g., [11, 82].

[13] In the relativistic quantum filed theories, where fields live in Minkowski spacetime, there are trustful conservation laws for any system of interacting fields.

[14] See, e.g., [59, 82].

This fact is very important because given a general Lorentzian spacetime modelling a given gravitational field according to *GRT* and which has a non constant scalar curvature in general there are not a sufficient number of Killing vector fields for formulating trustworthyl conservation laws for the energy-momentum and angular momentum of a given system of particles and fields. Many tentatives have been done in order to establish energy-momentum and angular momentum conservation laws in *GRT*. All tentatives are based on the fact that although the field $\boldsymbol{g}$ is part of the spacetime structure the 'geometry' must also have energy-momentum and angular momentum and thus it must exist mathematical objects that make the role of its 'energy-momentum tensor' and its angular momentum tensor, in order to warrant the conservation of the energy-momentum and angular momentum of the system composed of the gravitational and matter fields. However all tentatives resulted (until now) deceptive (unless some additional hypothesis are postulated for the Lorentzian spacetime structure, as e.g., that such structure is parallelizable) because all that have been attained was the association of a series of *pseudo-energy momentum tensors* to the gravitational field [15] and without additional commentaries from our part we quote here what the authors of [86] siad on this issue:

"*It is a shame to loose the special relativistic total energy conservation in General Relativity. Many of the tentatives to resurrect it are quite interesting, many are simply garbage*".

It can be shown that the non existence of trustworthy energy-momentum and angular momentum for the system of the gravitational field and the matter fields lead to serious inconsistencies. This has already been noted long ago by several scientists, in particular by Levi-Civita (already in 1919!) [58] and discussed by Logunov and collaborators [59] in the eights of the last century[16]. A modern discussion may be found in [2, 82, 69] and the continuous tentatives for finding a solution for the problem within *GRT* (including a today's popular approach called quasi-local energy) may be found in [95]. We will briefly discuss such an approach in Section 7 where we study the Hamiltonian formulation of *our* theory.

After more than 85 years of deceptive tentatives[17] many physicists think that *GRT* needs a revision where the main emphasis must be given to the fact that the gravitational field is a physical field in the *sense* of Faraday living and interacting with the other physical fields in Minkowski spacetime. However, the formulation of such a theory using as unique ingredients the geometric objects

[15] The first one to introduce an energy-momentum pseudo-tensor for the gravitational field was [23]. However, it is possible to introduce an infinity of distinct energy-momentum pseudo-tensors for the gravitational field in *GRT*. The mathematical reasons for this manifold possibility may be found, e.g., in [97, 82, 69].

[16] A simple way to understand the origin of the issue has already been clearly stated by Schröndinger, who in [92] observed that in a general Lorentzian spacetime you can not even define the momentum of a pair of particles when they are at events say, $\mathfrak{e}_1$ and $\mathfrak{e}_2$ because vectors at the tangent spaces $T_{\mathfrak{e}_1}M$ an $TM_{\mathfrak{e}_2}$ cannot be summed.

[17] Besides Logunov and collaborators we quote here also Feynmann [35], Schwinger [93], Weinberg [105], Rosen [85] and [43, 96, 40, 41].

of Minkowski spacetime and the correct representative of the gravitational field (the distortion field $\boldsymbol{h}$) living in that spacetime and in such way that the resulting theory becomes equivalent to *GRT* in some well defined sense resulted to be a nontrivial task.

In [80] (using the Clifford bundle formalism [82]) a theory of that kind, i.e., one where the gravitational field is described by a physical field living in Minkowski spacetime and has its dynamics described by a well defined Lagrangian density such that its equations of motion result equivalent to Einstein's field equations (at least for models of *GRT* where the manifold M can be taken as $\mathbb{R}^4$) has been proposed. In that theory it was suggested that the gravitational field $\boldsymbol{g}$ was to be interpreted as a deformation of the Minkowski metric $\boldsymbol{\eta}$ produced by a special $(1,1)$-extensor field $\boldsymbol{h} : \sec \bigwedge^1 T^*M \to \sec \bigwedge^1 T^*M$, said to be the distortion (or deformation) *tensor field.* However, in that opportunity those authors were not able to produce a mathematical formalism in which it was possible to work directly with $\boldsymbol{h}$, i.e., writing a Lagrangian density for that field and deducing its equation of motion.

The necessary mathematical formalism to do that job now exists, it is the *MECM* mentioned above. The *MECM* has its origin, by the best of our knowledge on some mathematical ideas first proposed by Hestenes and Sobczyc in their book "*Clifford Algebra to Geometrical Calculus*" [47]. Those ideas have been investigated in [28, 61, 91] and attained maturity in [26, 29, 30, 62, 63, 64, 65, 31, 32, 33, 34]. In particular in [65] the theory of derivatives of functionals of extensor fields necessary for the developments of the present paper has been introduced in a rigorous way. That crucial concept will be recalled (with several examples) in Section 3. Utilizing *MECM*, presented with enough details in Section 4 we will show that it is indeed possible to present a theory of the gravitational field as the theory of a field $\boldsymbol{h}$ living on Minkowski spacetime and describing a *plastic deformation* of the Lorentz vacuum. The Lorentz vacuum is in our view a real and very complex physical substance[18] which lives in an arena mathematically described by Minkowski spacetime[19]. When the Lorentz vacuum is in its ground state it is described by a trivial distortion field $\boldsymbol{h} = \mathrm{id}$, but when it is disturbed due to the presence of what we call matter fields it is described by a nontrivial distortion field $\boldsymbol{h}$ of the *plastic type* which as we shall see (Section 5) satisfies a well defined field equation, which are (Section 6) equivalent to Einstein field equation for $\boldsymbol{g}$ (at least for models of *GRT* where the manifold M can be taken as $\mathbb{R}^4$). In Section 6 we introduce also genuine energy-momentum and angular momentum conservation laws for the system composed of the matter and gravitational fields.

One of the features of the *general* mathematical formalism used in the present article is a formulation of a theory of covariant derivatives on manifolds, where some novel concepts appear, like for example the rotation extensor

[18]For some interesting ideas on the nature of such a substance see [56, 100, 101, 102].

[19]Eventually future experimental developments will show that this hypothesis must be modified, e.g., the substance describing the vacuum may live in a spacetime with more dimensions that Minkowski spacetime.

field Ω defined in a 4-dimensional vector manifold $U_o \subset U$, where U_o represents the points of a given $\mathcal{U} \subset M$ and U is called the canonical space.

Once U is constructed we introduce an Euclidean Clifford algebra denoted $\mathcal{C}\ell(U,\cdot)$ on $U_o \subset U$ whose *main purpose* is to define an algorithm to perform very sophisticated and indeed nontrivial calculations. Next we introduce on U_0 a constant Minkowski metric extensor field η and the distortion field $\boldsymbol{h}$ such that the extensor field $\boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}$ represents in U_o the metric tensor $\boldsymbol{g}$ in $\mathcal{U} \subset M$. Several different *MCGSS* are introduced having U_o as the first member of the triple and it is shown that some of those structures may be clearly interpreted as a distortion of another one through the action of the distortion extensor $\boldsymbol{h}$. In that way it will be shown that when $M \simeq \mathbb{R}^4$, in which case we may identify $\mathbb{R}^4 \simeq U_0 \simeq U$, the rotation extensor field that appears in the representative on U of the Levi-Civita connection of $\boldsymbol{g}$ can be written as a functional of $\boldsymbol{h}^{\clubsuit} = \boldsymbol{h}^{-1\dagger}$ and its *vector derivatives* $\cdot\partial\boldsymbol{h}^{\clubsuit}$ and $\cdot\partial\cdot\partial\boldsymbol{h}^{\clubsuit}$. This permits to write the usual Einstein-Hilbert Lagragian as a functional of $(\boldsymbol{h}^{\clubsuit}, \cdot\partial\boldsymbol{h}^{\clubsuit}, \cdot\partial\cdot\partial\boldsymbol{h}^{\clubsuit})$ and to find directly the equation of motion for $\boldsymbol{h}$, a result that for the best of our knowledge has not been obtained before in a consistent way. Indeed, it must be said that tentatives of using the rudiments of the multiform and extensor calculus to formulated a theory of the gravitational field as a gauge theory (but where the world gauge does not have the meaning it has in those field theories which use gauge fields defined as sections of some principal bundles) has been developed in [55, 21]. However, despite some very good ideas presented there, those works contains in our opinion some equivocated mathematical concepts which lead to inconsistencies. The gravitational theory in [55, 21] has been recently reviewed by [48] but it contains the same equivocated mathematical concepts. So, to clearly distinguish our theory from the one in [55, 21] we briefly discuss the main ingredients of that theory in Appendix F and point out explicitly what are those inconsistencies.

For completeness we also present in Appendix C the derivation of the equations of motion for a theory where the gravitational field is described by two independent fields, a distortion field $\boldsymbol{h}$ and a rotation field Ω. This is in line of ideas (first present in [46] where it is claimed that Ω is related to the source of spin.[20]. Appendix D contains a proof of the equivalence of two apparently very different expressions for the Einstein-Hilbert Lagrangian density. Appendix E recall the derivation of the field equations for the potential fields $\mathfrak{g}^{\alpha} = \boldsymbol{h}^{\dagger}(\boldsymbol{\vartheta}^{\alpha})$. Also, Appendix G presents (in order to illustrate once more the philosophy addopted in this paper) a model for the gravitational field of a point mass where that field is represented by the nonmetricity of a particular connection

Finally in Section 8 we present our conclusions.

[20]However, it must be said here that as it is clear from the the application of the Lagrangian formalism for multiform fields (including spinor fields) using the multiform and extensor calculus, as developed e.g., in [82] the source of spin is to be found in the antisymmetric part of the canonical energy-momentum tensor.

2 Multiforms and Extensors

Let $\mathbf{V}$ be a vector space ($\dim \mathbf{V} = n$) over the real field $\mathbb{R}$. The dual space of $\mathbf{V}$ is usually denoted by $\mathbf{V}^*$, but in what follows it will be denoted by V. The space of k-forms ($0 \leq k \leq n$) will be denoted by $\bigwedge^k V$ and we have the usual identifications: $\bigwedge^0 V = \mathbb{R}$ and $\bigwedge^1 V = V$.

2.1 Multiforms

A *formal* sum $X = X_0 + X_1 + \cdots + X_k + \cdots + X_n$, where $X_k \in \bigwedge^k V$ will be called a *nonhomogeneous multiform*. The set of all nonhomogeneous multiforms has a natural vector space structure[21] over $\mathbb{R}$ and will be denoted by $\bigwedge V$.

Let $\{\mathbf{e}_k\}$ and $\left\{\varepsilon^k\right\}$ be respectively basis of $\mathbf{V}$ and V where the second is the *dual* of the first, i.e., $\varepsilon^k(\mathbf{e}_j) = \delta^k_j$. Then $\{1, \varepsilon^{j_1}, \ldots, \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k}, \ldots, \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_n}\}$ is a basis for $\bigwedge V$, and since $\dim \mathbf{V} = n$ we have that $\dim \bigwedge V = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n} = 2^n$.

2.2 The k-Part Operator and Involutions

Let $0 \leq k \leq n$, the linear operator $\langle\rangle_k$ defined by $X \mapsto \langle X\rangle_k := X_k$ ($X_k \in \bigwedge^k V$) will be called the k-part operator and $\langle X\rangle_k$ is read as the *k-part* of X.

The k-forms are called *homogeneous multiforms of grade k* (or *k-homogeneous multiforms*). It is obvious that for any k-homogeneous multiform X we have: $\langle X\rangle_l = X$ if $l = k$, or $\langle X\rangle_l = 0$ if $l \neq k$. Also, any multiform may be expressed as a sum of its k-parts, i.e., $X = \sum\limits_{k=0}^{n} \langle X\rangle_k$.

The 0-forms $\langle X\rangle_0$ (i.e.,t he real numbers), the 1-forms $\langle X\rangle_1$, the 2-forms $\langle X\rangle_2$, etc...., are called scalars, forms, biforms, etc. .., and the n-forms $\langle X\rangle_n$ and the $(n-1)$-forms $\langle X\rangle_{n-1}$ are sometimes called pseudo-scalars and pseudo 1-forms [22].

We now introduce two fundamental involutions on $\bigwedge V$.

The linear operator $\hat{}$, defined by $X \mapsto \hat{X}$ such that $\left\langle \hat{X} \right\rangle_k = (-1)^k \langle X\rangle_k$, is called the *conjugation operator* and $\hat{X}$ *is read as the conjugate of* X.

The linear operator $\sim$, defined by $X \mapsto \widetilde{X}$ such that $\left\langle \widetilde{X} \right\rangle_k = (-1)^{\frac{1}{2}k(k-1)} \langle X\rangle_k$, is called the reversion operator and $\widetilde{X}$ is read as the reverse of X.

Both operators are involutions, i.e., $\hat{\hat{X}} = X$ and $\widetilde{\widetilde{X}} = X$.

[21] The sum of multiforms and the multiplicatin of multiforms by scalars are defined by: **(i)** if $X = X_0 + X_1 + \cdots + X_n$ e $Y = Y_0 + Y_1 + \cdots + Y_n$ then $X + Y = (X_0 + Y_0) + (X_1 + Y_1) + \cdots + (X_n + Y_n)$, **(ii)** if $\alpha \in R$ e $X = X_0 + X_1 + \cdots + X_n$ then $\alpha X = \alpha X_0 + \alpha X_1 + \cdots + \alpha X_n$.

[22] Such a nomenclature must be used with care, since it may lead to serious confusions if mislead with the concept of de Rham's pair and impair forms, those latter objects also called by some authors pseudo-forms or twisted forms. See [15] for a discussion of the issue.

2.3 Exterior Product

The exterior product (or Grassmann product) $\wedge : \bigwedge V \times \bigwedge V \to \bigwedge V$ is defined for arbitrary $X, Y \in \bigwedge V$ as the element $X \wedge Y \in \bigwedge V$ such that

$$\langle X \wedge Y\rangle_k = \sum_{j=1}^{k} \langle X\rangle_j \wedge \langle Y\rangle_{k-j}\,. \tag{11}$$

On the right side of Eq.(11) appears a sum of the exterior product of [23] j-form by a $(k-j)$-form. The exterior product (like the sum) is a an internal law in the space $\bigwedge V$. It satisfy distributive laws on the left and on the right, and the associative law $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$ for $X, Y, Z \in \bigwedge V$ and mist associativity, i.e., for $\alpha \in \mathbb{R}$ it holds, $\alpha(X \wedge Y) = (\alpha X \wedge Y) = (X \wedge \alpha Y)$. The first ones are consequence of the distributive laws of the exterior product of k-forms and the second ones may be shown to be true without difficulty. The linear space $\bigwedge V$ equipped with the exterior product is an associative algebra called the *exterior algebra of multiforms*.

We present now the main properties of the exterior product of multiforms.

(i) For all $\alpha, \beta \in \mathbb{R}$ and $X \in \bigwedge V$

$$\alpha \wedge \beta = \alpha\beta, \tag{12a}$$
$$\alpha \wedge X = X \wedge \alpha = \alpha X, \tag{12b}$$

(ii) For all $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$

$$X_j \wedge Y_k = (-1)^{jk} Y_k \wedge X_j. \tag{13}$$

(iii) For all $a \in \bigwedge^1 V$ and $X \in \bigwedge V$

$$a \wedge X = \hat{X} \wedge a. \tag{14}$$

(iv) For all $X, Y \in \bigwedge V$

$$(X \wedge Y)\hat{} = \hat{X} \wedge \hat{Y}, \tag{15a}$$
$$\widetilde{X \wedge Y} = \widetilde{Y} \wedge \widetilde{X}. \tag{15b}$$

[23]The exterior product of k-formas is defined as follows: **(i)** $\alpha, \beta \in \mathbb{R} : \alpha \wedge \beta = \alpha\beta$; **(ii)** $\alpha \in \mathbb{R}$ and $x \in \bigwedge^k V$ with $k \geq 1 : \alpha \wedge x = x \wedge \alpha = \alpha x$, **(iii)** if $X_j \in \bigwedge^j V$ and $X_j \in \bigwedge^k V$ with $j, k \geq 1$ then $X_j \wedge Y_k = \frac{(j+k)!}{j!k!} \mathcal{A}(x \otimes y)$ where $\mathcal{A}$ is the well known antisymmetrization operator of ordinary linear algebra. Keep in mind when reading texts on the subject that eventually another definition of the exterior product (not equivalent to the one used here) may be being used, as e.g., in [82].

2.4 The Scalar Canonical Product

Let us fix on $\mathbf{V}$ an arbitrary basis $\{\mathbf{b}_j\}$. A scalar product of multiforms $X, Y \in \bigwedge V$ may be defined by

$$X \cdot Y := \langle X \rangle_0 \langle Y \rangle_0 + \frac{1}{k!} \langle X \rangle_k (\mathbf{b}^{j_1}, \ldots, \mathbf{b}^{j_k}) \langle Y \rangle_k (b_{j_1}, \ldots, b_{j_k}), \tag{16}$$

where, in order to utilize Einstein's sum convention rule, we wrote $b_{j_i} := \mathbf{b}^{j_i}$, $i = 1, 2, ..., k$.

The product defined by Eq.(16) (obviously associated to the basis $\{\mathbf{b}_j\}$) is a well defined scalar product on the vector space $\bigwedge V$. It is symmetric ($X \cdot Y = Y \cdot X$), is linear, i.e., ($X \cdot (Y + Z) = X \cdot Y + X \cdot Z$), possess mist associativity ($(\alpha X) \cdot Y = X \cdot (\alpha Y) = \alpha(X \cdot Y)$) and is non degenerated (i.e., $X \cdot Y = 0$ for all X, implies $Y = 0$). Even more that product is positive definite, i.e., $X \cdot X \geq 0$ for all X, and $X \cdot X = 0$ iff $X = 0$.

The scalar product of multiforms which we just introduced will be called the *canonical scalar product*, despite the fact that for its definition it is necessary to choice an arbitrary basis $\{\mathbf{b}_j\}$ of $\mathbf{V}$.

Some important properties of the canonical scalar product are:

(i) For all scalars $\alpha, \beta \in \mathbb{R}$

$$\alpha \cdot \beta = \alpha\beta, \tag{17}$$

(ii) For all $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$

$$X_j \cdot Y_k = 0, \text{ if } j \neq k. \tag{18}$$

(iii) For any two simple k-forms, say $v_1 \wedge \ldots \wedge v_k$ and $w_1 \wedge \ldots \wedge w_k$, we have:

$$(v_1 \wedge \ldots \wedge v_k) \cdot (w_1 \wedge \ldots \wedge w_k) = \begin{vmatrix} v_1 \cdot w_1 & \cdots & v_1 \cdot w_k \\ \vdots & & \vdots \\ v_k \cdot w_1 & \cdots & v_k \cdot w_k \end{vmatrix}, \tag{19}$$

where $|\,|$ denote here, the classical determinant.

(iv) For all $X, Y \in \bigwedge V$

$$\hat{X} \cdot Y = X \cdot \hat{Y}, \tag{20a}$$

$$\widetilde{X} \cdot Y = X \cdot \widetilde{Y}. \tag{20b}$$

(v) Let $(\{\varepsilon^j\}, \{\varepsilon_j\})$ be a pair of reciprocal bases for V (i.e., $\varepsilon^j \cdot \varepsilon_k = \delta^j_k$). We can then construct exactly two natural basis for the space $\bigwedge V$, respectively

$$\{1, \varepsilon^{j_1}, \ldots, \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k}, \ldots, \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_n}\}, \tag{21a}$$

and

$$\{1, \varepsilon_{j_1}, \ldots, \varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_k}, \ldots, \varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_n}\}. \tag{21b}$$

Then, any $X \in \bigwedge V$ may be expressed using those basis as :

$$X = X \cdot 1 + \sum_{k=1}^{n} \frac{1}{k!} X \cdot (\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k})(\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_k}), \tag{22a}$$

$$= X \cdot 1 + \sum_{k=1}^{n} \frac{1}{k!} X \cdot (\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_k})(\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k}), \tag{22b}$$

or in a more compact notation using the collective index[24] J,

$$X = \sum_J \frac{1}{\nu(J)!} (X \cdot \varepsilon^J) \varepsilon_J = \sum_J \frac{1}{\nu(J)!} (X \cdot \varepsilon_J) \varepsilon^J. \tag{23}$$

2.5 Canonical Contract Products

We now introduce two other important products on $\bigwedge V$ by utilizing only the properties of the canonical scalar product.

Given $X, Y \in \bigwedge V$ the *canonical left contraction* $(X, Y) \mapsto X \lrcorner Y$ is defined by

$$(X \lrcorner Y) \cdot Z := Y \cdot (\widetilde{X} \wedge Z), \tag{24}$$

for arbitrary $Z \in \bigwedge V$.

Given $X, Y \in \bigwedge V$ the canonical right contraction $(X, Y) \mapsto X \llcorner Y$ is defined by

$$(X \llcorner Y) \cdot Z := X \cdot (Z \wedge \widetilde{Y}), \tag{25}$$

for arbitrary $Z \in \bigwedge V$.

The space $\bigwedge V$ equipped with anyone of those canonical contractions $\lrcorner$ or $\llcorner$ possess a natural structure of a non associative algebra. The triple $(\bigwedge V, \lrcorner, \llcorner)$ will be called the canonical interior algebra of multiforms

Some important properties of $\lrcorner$ and $\llcorner$ are:

[24] The collective index J take values over the follwing set of indices: $\emptyset$ (the null set), $j_1, \ldots, j_1 \ldots j_k, \ldots, j_1 \ldots j_n$. The symbol ε^J denots then a scalar basis, $\varepsilon^{\emptyset} = 1$, a basis of 1-forms $\varepsilon^{j_1}, \ldots$, and $\varepsilon^{j_1 \cdots j_k} = \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k}$ abasis for $\bigwedge^k V$. We will use also when convenient the notation $\varepsilon^{j_1 \cdots j_n} = \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_n}$. Finally we register that $\nu(J)$ is the number of indices, defined by $\nu(J) = 0, 1, \ldots, k, \ldots, n$ with J taking values over the set just introduced.

(i) For all $\alpha, \beta \in \mathbb{R}$ and $X \in \bigwedge V$

$$\alpha \lrcorner \beta = \alpha \llcorner \beta = \alpha\beta, \tag{26a}$$

$$\alpha \lrcorner X = X \llcorner \alpha = \alpha X, \tag{26b}$$

(ii) For all $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$

$$X_j \lrcorner Y_k = 0, \text{ if } j > k, \tag{27}$$

$$X_j \llcorner Y_k = 0, \text{ if } j < k. \tag{28}$$

(iii) For all $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$, with $j \leq k$, we have that $X_j \lrcorner Y_k \in \bigwedge^{k-j} V$, $Y_k \llcorner X_j \in \bigwedge^{k-j} V$ and

$$X_j \lrcorner Y_k = (-1)^{j(k-j)} Y_k \llcorner X_j. \tag{29}$$

(iv) For all $X_k, Y_k \in \bigwedge^k V$

$$X_k \lrcorner Y_k = X_k \llcorner Y_k = \widetilde{X_k} \cdot Y_k = X_k \cdot \widetilde{Y_k}. \tag{30}$$

(v) For all $a \in \bigwedge^1 V$ and $X \in \bigwedge V$

$$a \lrcorner X = -\hat{X} \llcorner a. \tag{31}$$

(vi) For all $a \in \bigwedge^1 V$ and $X, Y \in \bigwedge V$

$$a \lrcorner (X \wedge Y) = (a \lrcorner X) \wedge Y + \hat{X} \wedge (a \lrcorner Y). \tag{32}$$

(vii) For all $X, Y, Z \in \bigwedge V$

$$X \lrcorner (Y \lrcorner Z) = (X \wedge Y) \lrcorner Z, \tag{33a}$$

$$(X \llcorner Y) \llcorner Z = X \llcorner (Y \wedge Z). \tag{33b}$$

2.6 The Canonical Clifford Product

We define now on $\bigwedge V$ the *canonical Clifford product* (or *geometrical product*) utilizing the canonical contractions and the exterior product. For any $X, Z \in \bigwedge V$ the Clifford product of X by Y will be denoted by juxtaposition of symbols and obeys the following axiomatic:

(i) For all $\alpha \in \mathbb{R}$ and $X \in \bigwedge V$

$$\alpha X = X\alpha \tag{34}$$

(ii) For all $a \in \bigwedge^1 V$ and any $X \in \bigwedge V$

$$aX = a \lrcorner X + a \wedge X, \tag{35a}$$

$$Xa = X \llcorner a + X \wedge a. \tag{35b}$$

(iii) For all $X, Y, Z \in \bigwedge V$

$$X(YZ) = (XY)Z. \tag{36}$$

The canonical Clifford product is distributive and associative (precisely axiom **(iii)**). Its distributive law follows directly from the corresponding distributive laws of the canonical contractions and of the exterior product.

The linear space $\bigwedge V$ equipped with the canonical Clifford product is a (real) associative algebra. It will be called here the canonical Clifford algebra of multiforms *and denoted by* $\mathcal{C}\ell(V, \cdot)$.

We list now the main properties of the canonical Clifford product.

(i) For all $X, Y \in \bigwedge V$

$$(XY)^\wedge = \hat{X}\ \hat{Y}, \tag{37a}$$

$$\widetilde{XY} = \widetilde{Y}\widetilde{X}. \tag{37b}$$

(ii) For all $a \in \bigwedge^1 V$ and $X \in \bigwedge V$

$$a \lrcorner X = \frac{1}{2}(aX - \hat{X}\ a). \tag{38a}$$

$$a \wedge X = \frac{1}{2}(aX + \hat{X}a). \tag{38b}$$

(iii) For all $X, Y \in \bigwedge V$

$$X \cdot Y = \left\langle \widetilde{X}Y \right\rangle_0 = \left\langle X\widetilde{Y} \right\rangle_0. \tag{39}$$

(iv) Let $\tau \in \bigwedge^n V$, $a \in \bigwedge^1 V$ and $X \in \bigwedge V$, then

$$\tau(a \wedge X) = (-1)^{n-1} a \lrcorner (\tau X). \tag{40}$$

This notable formula is sometimes referred in the mathematical literature as the *duality identity.*

2.7 Extensors

2.7.1 The Space $extV$

A linear mapping between two arbitrary parts[25] $\overset{\diamond}{\bigwedge}_1 V, \overset{\diamond}{\bigwedge}_2 V$ of $\bigwedge V$, i.e., $t : \overset{\diamond}{\bigwedge}_1 V \to \overset{\diamond}{\bigwedge}_2 V$ such that for all $\alpha, \alpha' \in \mathbb{R}$ and $X, X' \in \overset{\diamond}{\bigwedge}_1 V$ we have

$$t(\alpha X + \alpha' X') = \alpha t(X) + \alpha' t(X'), \tag{41}$$

will be called an extensor over V. The set of all extensor over V whose domain and codomain are $\overset{\diamond}{\bigwedge}_1 V$ and $\overset{\diamond}{\bigwedge}_2 V$ possess a natural structure of vector space over $\mathbb{R}$.

When $\overset{\diamond}{\bigwedge}_1 V = \overset{\diamond}{\bigwedge}_2 V = \bigwedge V$ the space of extensors $t : \bigwedge V \to \bigwedge V$ will be denoted by $extV$.

2.7.2 The Space (p,q)-$extV$ of the (p,q)-Extensors

Let p, q two integers with $0 \leq p, q \leq n$. An extensor with domain $\bigwedge^p V$ and codomain $\bigwedge^q V$ is said to be a *(p,q)-extensor over V*. The real vector space of the (p,q)-extensors will be denoted (p,q)-$extV$. If $\dim \mathbf{V} = n$, then $\dim((p,q)$-$extV) = \binom{n}{p}\binom{n}{q}$.

2.7.3 The Adjoint Operator

Let $(\{\varepsilon^j\}, \{\varepsilon_j\})$ be a pair of arbitrary reciprocal basis for V (i.e., $\varepsilon^j \cdot \varepsilon_i = \delta_i^j$). The linear operator acting on the space (p,q)-$extV \ni t \mapsto t^\dagger \in (q,p)$-$extV$ such that

$$t^\dagger(X) := \frac{1}{p!}(t(\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_p}) \cdot X)\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_p} \tag{42a}$$

$$= \frac{1}{p!}(t(\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_p}) \cdot X)\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_p}, \tag{42b}$$

is called the *adjoint operator* and $t^\dagger$ is read as the adjoint of t.

The adjoint operator is well defined since the sums appearing in Eqs.(42a),(42b) do not depend on the choice of the pair of reciprocal basis $(\{\varepsilon^j\}, \{\varepsilon_j\})$.

We give now some important properties of the adjoint operator.

(i) The adjoint operator is involutive, i.e.,

$$(t^\dagger)^\dagger = t. \tag{43}$$

(ii) Let $t \in (p,q)$-$extV$. Then for all $X \in \bigwedge^p V$, $Y \in \bigwedge^q V$ we have that

$$t(X) \cdot Y = X \cdot t^\dagger(Y). \tag{44}$$

[25] A direct sum of any vector spaces $\bigwedge^k V$ is clearly a subspace of $\bigwedge V$ and is said to be a *part* of $\bigwedge V$. A convenient notation for a part of $\bigwedge V$ is $\overset{\diamond}{\bigwedge} V$.

(iii) Let $t \in (q,r)$-$extV$ and $u \in (p,q)$-$extV$. The extensor $t \circ u$ (composition of u with t), which clearly belongs to (p,r)-$extV$, satisfies

$$(t \circ u)^{\dagger} = u^{\dagger} \circ t^{\dagger}. \tag{45}$$

2.7.4 $(1,1)$-Extensors

Symmetric and Antisymmetric parts of $(1,1)$-Extensors An extensor $t \in (1,1)$-$extV$ is said to be *adjoint symmetrical* (respectively *adjoint antisymmetric*) iff $t = t^{\dagger}$ (respectively $t = -t^{\dagger}$).

The following result is important. For any $t \in (1,1)$-$extV$, there exist exactly two $(1,1)$-extensors over V**,** say t_+ and t_-, such that t_+ is adjoint symmetric (i.e., $t_+ = t_+^{\dagger}$) and t_- is adjoint antisymmetric (i.e., $t_- = -t_-^{\dagger}$) and the following decomposition is valid:

$$t(a) = t_+(a) + t_-(a). \tag{46}$$

Those $(1,1)$-extensors are given by

$$t_{\pm}(a) = \frac{1}{2}(t(a) \pm t^{\dagger}(a)). \tag{47}$$

The extensors t_+ and t_- are called the adjoint symmetric and the adjoint antisymmetric parts *of* t.

The extension of $(1,1)$-Extensors Let $(\{\varepsilon^j\},\{\varepsilon_j\})$ be an arbitrary pair of reciprocal basis for$V$ (i.e., $\varepsilon^j \cdot \varepsilon_i = \delta_i^j$) and let $t \in (1,1)$-$extV$. The *extension operator* is the linear mapping

$$\begin{aligned} &\underline{\quad} : (1,1)\text{-}extV \to extV, \\ &t \mapsto \underline{t}, \end{aligned} \tag{48}$$

such that

$$\underline{t}(X) := 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!}((\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k}) \cdot X) t(\varepsilon_{j_1}) \wedge \ldots \wedge t(\varepsilon_{j_k}) \tag{49}$$

$$= 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!}((\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_k}) \cdot X) t(\varepsilon^{j_1}) \wedge \ldots \wedge t(\varepsilon^{j_k}). \tag{50}$$

In what follows we say that $\underline{t}$ is the *extension of* t [26].

The extension operator is well defined since the sums appearing in Eqs.(49) and (50) do not depend on the pair $(\{\varepsilon^j\},\{\varepsilon_j\})$. Moreover the extension operator preserves the graduation, i.e., if $X \in \overset{k}{\bigwedge} V$, then $\underline{t}(X) \in \overset{k}{\bigwedge} V$.

We present now some important properties of the $(1,1)$-extensors

[26] Some authors call $\underline{t}$ the *exterior algebra extension* of t.

(i) Let $t \in (1,1)$-$extV$, for all $\alpha \in \mathbb{R}$, $v \in \bigwedge V$ and $v_1, \ldots, v_k \in \bigwedge V$ we have that

$$\underline{t}(\alpha) = \alpha, \tag{51a}$$
$$\underline{t}(v) = t(v), \tag{51b}$$
$$\underline{t}(v_1 \wedge \ldots \wedge v_k) = t(v_1) \wedge \ldots \wedge t(v_k). \tag{51c}$$

The last property possess an immediate corollary,

$$\underline{t}(X \wedge Y) = \underline{t}(X) \wedge \underline{t}(Y), \tag{52}$$

for all $X, Y \in \bigwedge V$.

(ii) For all $t, u \in (1,1)$-$extV$ the following identity holds;

$$\underline{t \circ u} = \underline{t} \circ \underline{u}. \tag{53}$$

(iii) Let $t \in (1,1)$-$extV$ with inverse $t^{-1} \in (1,1)$-$extV$ (i.e., $t \circ t^{-1} = t^{-1} \circ t = i_d$, where $i_d \in (1,1)$-$extV$ is the identity extensor), then

$$(\underline{t})^{-1} = \underline{(t^{-1})}, \tag{54}$$

and we shall use the symbol $\underline{t}^{-1}$ to denote both extensors $(\underline{t})^{-1}$ and $\underline{(t^{-1})}$.

(iv) For any $(1,1)$-extensor over V, the extension operator commutes with the adjoint operator, i.e.,

$$\underline{(t^{\dagger})} = (\underline{t})^{\dagger}, \tag{55}$$

and we shall use the symbol $\underline{t}^{\dagger}$ to denote both extensors $\underline{(t^{\dagger})}$ and $(\underline{t})^{\dagger}$.

(v) Let $t \in (1,1)$-$extV$, for all $X, Y \in \bigwedge V$ we have

$$X \lrcorner \underline{t}(Y) = \underline{t}(\underline{t}^{\dagger}(X) \lrcorner Y). \tag{56}$$

The Characteristic Scalars $tr[t]$ **and** $\det[t]$ Let $(\{\varepsilon^j\}, \{\varepsilon_j\})$ be an arbitrary pair of reciprocal basis for V (i.e., $\varepsilon^j \cdot \varepsilon_i = \delta_i^j$). The *trace mapping*

$$\begin{aligned} \mathrm{tr} : (1,1)\text{-}extV &\to \mathbb{R}, \\ t &\mapsto tr[t] \end{aligned} \tag{57}$$

is such that

$$\mathrm{tr}[t] := t(\varepsilon^j) \cdot \varepsilon_j = t(\varepsilon_j) \cdot \varepsilon^j. \tag{58}$$

Observe that tr does not depend on the pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$ used in Eq.(58). We have immediately that for any $t \in (1,1)$-$extV$,

$$\mathrm{tr}[t^{\dagger}] = \mathrm{tr}[t]. \tag{1.42}$$

The *determinant mapping*

$$\begin{aligned} \det[t] : (1,1)\text{-}extV &\to \mathbb{R}, \\ t &\mapsto \det[t], \end{aligned}$$

is such that

$$\det[t] := \frac{1}{n!}\underline{t}(\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_n}) \cdot (\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_n}) \tag{59a}$$

$$= \frac{1}{n!}\underline{t}(\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_n}) \cdot (\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_n}). \tag{59b}$$

Like the trace the determinant also does not depend on the pair of arbitrary reciprocal basis $(\{\varepsilon^j\}, \{\varepsilon_j\})$.

By using the combinatorial formulas $v^{j_1} \wedge \ldots \wedge v^{j_n} = \epsilon^{j_1 \cdots j_n} v^1 \wedge \ldots \wedge v^n$ and $v_{j_1} \wedge \ldots \wedge v_{j_n} = \epsilon_{j_1 \ldots j_n} v_1 \wedge \ldots \wedge v_n$, where $\epsilon^{j_1 \cdots j_n}$ and $\epsilon_{j_1 \cdots j_n}$ are symbols of permutation[27] of order n and $v^1, \ldots, v^n$ and $v_1, \ldots, v_n$ are 1-forms, we can write formulas more convenient for $\det t$. Indeed,we have:

$$\det[t] = \underline{t}(\varepsilon^1 \wedge \ldots \wedge \varepsilon^n) \cdot (\varepsilon_1 \wedge \ldots \wedge \varepsilon_n) = \underline{t}(\overline{\varepsilon}) \cdot \underline{\varepsilon} \tag{60a}$$

$$= \underline{t}(\varepsilon_1 \wedge \ldots \wedge \varepsilon_n) \cdot (\varepsilon^1 \wedge \ldots \wedge \varepsilon^n) = \underline{t}(\underline{\varepsilon}) \cdot \overline{\varepsilon}, \tag{60b}$$

where we used the short notations: $\overline{\varepsilon} = \varepsilon^1 \wedge \ldots \wedge \varepsilon^n$ and $\underline{\varepsilon} = \varepsilon_1 \wedge \ldots \wedge \varepsilon_n$.

The mapping det possess the following important properties.

(i) For all $t \in (1,1)$-$extV$,

$$\det[t^\dagger] = \det[t]. \tag{61}$$

(ii) Let $t \in (1,1)$-$extV$, for any $\tau \in \bigwedge V$ we have

$$\underline{t}(\tau) = \det[t]\tau. \tag{62}$$

(iii) For all $t, u \in (1,1)$-$extV$,

$$\det[t \circ u] = \det[t] \circ \det[u]. \tag{63}$$

(iv) Let $t \in (1,1)$-$extV$ with inverse $t^{-1} \in (1,1)$-$extV$ (i.e., $t \circ t^{-1} = t^{-1} \circ t = i_d$, where $i_d \in (1,1)$-$extV$ is the identity extensor). Then,

$$\det[t^{-1}] = (\det[t])^{-1}. \tag{64}$$

We shall use the notation $\det^{-1}[t]$ to denote both $\det[t^{-1}]$ and $(\det[t])^{-1}$.

(v) If $t \in (1,1)$-$extV$ is non degenerated (i.e., $\det[t] \neq 0$), then its inverse $t^{-1} \in (1,1)$-$extV$ exists and is given by

$$t^{-1}(a) = \det{}^{-1}[t]\underline{t}^\dagger(a\tau)\tau^{-1}, \tag{65}$$

where $a \in \overset{1}{\bigwedge} V$ and $\tau \in \overset{n}{\bigwedge} V$, $\tau \neq 0$.

[27]The permutation symbols of order n are defined by

$$\epsilon^{j_1 \cdots j_n} = \epsilon_{j_1 \cdots j_n} = \begin{cases} 1, & \text{if } j_1 \ldots j_n \text{ is an even permutation of } 1, \ldots, n \\ -1, & \text{if } j_1 \ldots j_n \text{ is an odd permutation of } 1, \ldots, n \\ 0, & \text{in all other cases.} \end{cases}.$$

Moreover, recall that $\epsilon^{j_1 \cdots j_n} \epsilon_{j_1 \cdots j_n} = n!$

The Characteristic Biform Mapping bif Let $(\{\varepsilon^j\}, \{\varepsilon_j\})$ be an arbitrary pair of reciprocal basis for V (i.e., $\varepsilon^j \cdot \varepsilon_i = \delta_i^j$). The *biform mapping*

$$\text{bif} : (1,1)\text{-}extV \to \bigwedge^2 V,$$

is such that

$$\text{bif}[t] := t(\varepsilon^j) \wedge \varepsilon_j = t(\varepsilon_j) \wedge \varepsilon^j. \tag{66}$$

Note that bif$[t]$ is a 2-form characteristic of the extensor t (since it does not depend on the pair of reciprocal basis) $(\{\varepsilon^j\}, \{\varepsilon_j\})$, and bif$[t]$ is read as the *biform* of t.

We give now some important properties of bif mapping:

(i) Let $t \in (1,1)$-$extV$, then

$$\text{bif}[t^\dagger] = -\text{bif}[t]. \tag{67}$$

(ii) The antisymmetric adjoint of any $t \in (1,1)$-$extV$ may be factored by the following formula

$$t_-(a) = \frac{1}{2}\text{bif}[t] \times a, \tag{68}$$

where $\times$ means here the canonical commutator defined for any $A, B \in \bigwedge V$ by

$$A \times B := \frac{1}{2}(AB - BA). \tag{69}$$

2.7.5 The Generalization Operator of $(1,1)$-Extensors

Let $(\{\varepsilon^j\}, \{\varepsilon_j\})$ be an arbitrary pair of reciprocal basis for V (i.e., $\varepsilon^j \cdot \varepsilon_i = \delta_i^j$). The linear operator whose domain is the space $(1,1)$-$extV$ and whose codomain is the space $extV$, given by

$$\begin{aligned}(1,1)\text{-}extV \ni t \mapsto T \in extV, \\ T(X) = t(\varepsilon^j) \wedge (\varepsilon_j \lrcorner X) = t(\varepsilon_j) \wedge (\varepsilon^j \lrcorner X),\end{aligned} \tag{70}$$

is called the *generalization operator*, and T is read as the *generalized of* t.

The generalization operator is well defined since the sum in Eq.(70) does not depend on $(\{\varepsilon^j\}, \{\varepsilon_j\})$. The generalization operators preserves the grade, i.e., if $X \in \bigwedge^k V$, then $T(X) \in \bigwedge^k V$.

We list some important properties of the generalization of a $(1,1)$-extensor t:

(i) For all $\alpha \in \mathbb{R}$ and $v \in \bigwedge^1 V$,

$$T(\alpha) = 0, \tag{71a}$$

$$T(v) = t(v). \tag{71b}$$

(ii) For all $X, Y \in \bigwedge V$ it is:

$$T(X \wedge Y) = T(X) \wedge Y + X \wedge T(Y) \tag{72}$$

(iii) The generalization operator commutes with the adjoint operator. Then $T^{\dagger}$ denotes both the adjoint of the generalized and the generalized of the *adjoint.*

(iv) The antisymmetric part of the adjoint of the generalized coincides with the generalized of the adjoint antisymmetric part of t and can then be factored as

$$T_{-}(X) = \frac{1}{2}\text{bif}[t] \times X, \tag{73}$$

for all $X \in \bigwedge V$.

Normal $(1,1)$**-Extensors** A extensor $t \in (1,1)$-$extV$ is said to be *normal* if its adjoint symmetric and antisymmetric parts commutes, i.e., for any $a \in V$,

$$t_{+}(t_{-}(a)) = t_{-}(t_{+}(a)). \tag{74}$$

We can show without difficulties [47] that Eq.(74) is equivalent to any one of the following two equations

$$\begin{aligned} \underline{t}(\underline{t}^{\dagger}(A)) &= \underline{t}^{\dagger}(\underline{t}(A)), \\ \langle \underline{t}(A)\underline{t}(B)\rangle_0 &= \langle \underline{t}^{\dagger}(A)\underline{t}^{\dagger}(B)\rangle_0, \end{aligned} \tag{75}$$

for any $A, B \in \bigwedge V$

2.8 The Metric Clifford Algebra $\mathcal{C}\ell(V, \boldsymbol{g})$

A $(1,1)$-extensor over V, say $\boldsymbol{g}$ which is symmetric adjoint (i.e., $\boldsymbol{g} = \boldsymbol{g}^{\dagger}$) and non degenerated (i.e., $\det \boldsymbol{g} \neq 0$) is said to be a metric extensor on V. Under these conditions $\boldsymbol{g}^{-1}$ exists and is also a metric extensor on V, and as we are going to see both $\boldsymbol{g}$ and $\boldsymbol{g}^{-1}$ have equally important roles in our theory. The reason is that we are going to use $\boldsymbol{g}$ to represent the metric tensor $\mathbf{g} \in T_2^0\mathbf{V}$ whereas $\boldsymbol{g}^{-1}$ will represent $\mathsf{g} \in T_0^2\mathbf{V}$. This is done as follows. Given an arbitrary basis $\{\mathbf{e}_i\}$ of $\mathbf{V}$ and the corresponding dual basis $\{\varepsilon^i\}$ de V and the reciprocal basis of $\{\varepsilon^i\}$, i.e., $\varepsilon_i \cdot \varepsilon^j = \delta_i^j$, if $\mathbf{g}(\mathbf{e}_i, \mathbf{e}_j) = g_{ij}$ and $\mathsf{g}(\varepsilon^i, \varepsilon^j) = g^{ij}$ with $g_{ij}g^{jk} = \delta_i^k$ then:

$$\begin{aligned} \boldsymbol{g}^{-1}(\varepsilon^i) \cdot \varepsilon^j &= g^{ij}, \\ \boldsymbol{g}(\varepsilon_i) \cdot \varepsilon_j &= g_{ij}. \end{aligned} \tag{76}$$

2.8.1 The Metric Scalar Products

Given a metric extensor $\boldsymbol{g} \in (1,1)\text{-}extV$ we may define two new scalar products on $\bigwedge V$, respectively $\underset{g}{\cdot}$ and $\underset{g^{-1}}{\cdot}$ such that:

$$X \underset{g}{\cdot} Y := \underline{\boldsymbol{g}}(X) \cdot Y, \tag{77}$$

$$X \underset{g^{-1}}{\cdot} Y := \underline{\boldsymbol{g}}^{-1}(X) \cdot Y \tag{78}$$

Both $\underset{g}{\cdot}$ and $\underset{g^{-1}}{\cdot}$ will be called a *metric scalar product.*

The mappings $\underset{g}{\cdot}$ and $\underset{g^{-1}}{\cdot}$ are indeed well defined scalar products on $\bigwedge V$. The scalar product $\underset{g}{\cdot}$ (and obviously also $\underset{g^{-1}}{\cdot}$) is symmetric (i.e., $X \underset{g}{\cdot} Y = Y \underset{g}{\cdot} X$), is linear (i.e., $X \underset{g}{\cdot} (Y + Z) = X \underset{g}{\cdot} Y + X \underset{g}{\cdot} Z$), possess mist associativity (i.e., $(\alpha X) \underset{g}{\cdot} Y = X \underset{g}{\cdot} (\alpha Y) = \alpha(X \underset{g}{\cdot} Y)$, $\forall \alpha \in \mathbb{R}$)) and is non degenerated (i.e., $X \underset{g}{\cdot} Y = 0$ for all X, implies $Y = 0$). In what follows we list the main properties of $\underset{g}{\cdot}$, since $\underset{g^{-1}}{\cdot}$ obey similar properties.

(i) For $\alpha, \beta \in \mathbb{R}$

$$\alpha \underset{g}{\cdot} \beta = \alpha\beta, \tag{79}$$

(ii) For all $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$

$$X_j \underset{g}{\cdot} Y_k = 0, \text{ if } j \neq k. \tag{80}$$

(iii) For all simple k-forms $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$ and $w_1 \wedge \ldots \wedge w_k \in \bigwedge^k V$, we have

$$(v_1 \wedge \ldots \wedge v_k) \underset{g}{\cdot} (w_1 \wedge \ldots \wedge w_k) = \begin{vmatrix} v_1 \underset{g}{\cdot} w_1 & \ldots & v_1 \underset{g}{\cdot} w_k \\ \vdots & & \vdots \\ v_k \underset{g}{\cdot} w_1 & \ldots & v_k \underset{g}{\cdot} w_k \end{vmatrix}, \tag{81}$$

where as before $|\,|$ denotes the classical determinant of the matrix with the entries $v_i \underset{g}{\cdot} w_j$.

(iv) For all $X, Y \in \bigwedge V$

$$\hat{X} \underset{g}{\cdot} Y = X \underset{g}{\cdot} \hat{Y}. \tag{82a}$$

$$\widetilde{X} \underset{g}{\cdot} Y = X \underset{g}{\cdot} \widetilde{Y}. \tag{82b}$$

2.8.2 The Metric Left and Right Contraction

The metric left and right contractions of $X, Y \in \bigwedge V$ are given by

$$X \underset{g}{\lrcorner} Y := \underline{\boldsymbol{g}}(X) \lrcorner Y, \tag{83}$$

$$X \underset{g}{\llcorner} Y := X \llcorner \underline{\boldsymbol{g}}(Y). \tag{84}$$

The linear space $\bigwedge V$ equipped with any one of those contractions has a structure of non associative algebra, and $(\bigwedge V, \underset{g}{\lrcorner}, \underset{g}{\llcorner})$ will be called metric interior algebra of multiforms.

The most important properties of the metric left and right contractions are:

(i) For all $X, Y,\ Z \in \bigwedge V$,

$$(X \underset{g}{\lrcorner} Y) \underset{g}{\cdot} Z = Y \underset{g}{\cdot} (\widetilde{X} \wedge Z), \tag{85a}$$

$$(X \underset{g}{\llcorner} Y) \underset{g}{\cdot} Z = X \underset{g}{\cdot} (Z \wedge \widetilde{Y}). \tag{85b}$$

(ii) For all $\alpha, \beta \in \mathbb{R}$ and $X \in \bigwedge V$

$$\alpha \underset{g}{\lrcorner} \beta = \alpha \underset{g}{\llcorner} \beta = \alpha\beta, \tag{86a}$$

$$\alpha \underset{g}{\lrcorner} X = X \underset{g}{\llcorner} \alpha = \alpha X, \tag{86b}$$

(iii) For all $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$

$$X_j \underset{v}{\lrcorner} Y_k = 0, \text{ if } j > k. \tag{87a}$$

$$X_j \underset{g}{\llcorner} Y_k = 0, \text{ if } j < k. \tag{87b}$$

(iv) For all $X_j \in \bigwedge^j V$ and $Y_k \in \bigwedge^k V$, with $j \leq k$, we have: $X_j \underset{g}{\lrcorner} Y_k \in \bigwedge^{k-j} V$ and $Y_k \underset{g}{\llcorner} X_j \in \bigwedge^{k-j} V$ and

$$X_j \underset{g}{\lrcorner} Y_k = (-1)^{j(k-j)} Y_k \underset{g}{\llcorner} X_j. \tag{88}$$

(v) For all $X_k, Y_k \in \bigwedge^k V$

$$X_k \underset{g}{\lrcorner} Y_k = X_k \underset{g}{\llcorner} Y_k = \widetilde{X}_k \underset{g}{\cdot} Y_k = X_k \underset{g}{\cdot} \widetilde{Y}_k. \tag{89}$$

(vi) For all $a \in \bigwedge^1 V$ and $X \in \bigwedge V$

$$a \underset{g}{\lrcorner} X = -\hat{X} \underset{g}{\llcorner} a. \tag{90}$$

(vii) For all $a \in \bigwedge^1 V$ and $X, Y \in \bigwedge V$

$$a \underset{g}{\lrcorner} (X \wedge Y) = (a \underset{g}{\lrcorner} X) \wedge Y + \hat{X} \wedge (a \underset{g}{\lrcorner} Y). \tag{91}$$

2.8.3 The Metric Clifford Product

The metric Clifford product of two arbitrary elements $X, Y \in \bigwedge V$ will be written $X \underset{g}{} Y$ and satisfies the following axiomatic:

(i) For all $\alpha \in \mathbb{R}$ and $X \in \bigwedge V$

$$\alpha \underset{g}{} X = X \underset{g}{} \alpha = \alpha X,$$

(ii) For all $a \in \bigwedge^1 V$ and $X \in \bigwedge V$

$$\begin{aligned} a \underset{g}{} X &= a \underset{g}{\lrcorner} X + a \wedge X, \\ X \underset{g}{} a &= X \underset{g}{\llcorner} a + X \wedge a. \end{aligned}$$

(iii) For all $X, Y, Z \in \bigwedge V$

$$X \underset{g}{} (Y \underset{g}{} Z) = (X \underset{g}{} Y) \underset{g}{} Z.$$

The metric Clifford product is distributive and associative. Its distributive law follows from the correspondent distributive laws satisfied by the left and right contractions and the exterior product. Its associative law is just the axiom **(iii)**.

$\bigwedge V$ equipped with a metric Clifford product is an associative algebra, called a *metric Clifford algebra over* V. It will be denoted by $\mathcal{C}\ell(V, \boldsymbol{g})$. In a totally similar way we may define the algebra $\mathcal{C}\ell(V, \boldsymbol{g}^{-1})$.

We conclude this section by listing some useful properties of the metric Clifford product which will be used in the calculations of the following sections.

(i) For all $a \in \bigwedge^1 V$ and $X, Y, Z \in \bigwedge V$:

$$\begin{aligned} \widehat{X \underset{g}{} Y} &= \hat{X} \underset{g}{} \hat{Y}, \\ \widetilde{X \underset{g}{} Y} &= \widetilde{X} \underset{g}{} \widetilde{Y}. \\ a \underset{g}{\lrcorner} X &= \frac{1}{2}(a \underset{g}{} X - \hat{X} \underset{g}{} a), \\ a \wedge X &= \frac{1}{2}(a \underset{g}{} X + \hat{X} \underset{\boldsymbol{g}}{} a). \end{aligned} \tag{92}$$

$$
\begin{aligned}
X \underset{g}{\cdot} Y &= \left\langle \widetilde{X} \underset{g}{\,} Y \right\rangle_0 = \left\langle X \underset{g}{\,} \widetilde{Y} \right\rangle_0 . \\
X \underset{g}{\lrcorner} (Y \underset{g}{\lrcorner} Z) &= (X \wedge Y) \underset{g}{\lrcorner} Z, \\
(X \underset{g}{\llcorner} Y) \underset{g}{\llcorner} Z &= X \underset{g}{\llcorner} (Y \wedge Z).
\end{aligned} \tag{93}
$$

(ii) Let $\tau \in \bigwedge^n V$ then for all $a \in \bigwedge^1 V$ and $X \in \bigwedge V$:

$$
\tau \underset{g}{\,} (a \wedge X) = (-1)^{n-1} a \underset{g}{\lrcorner} (\tau \underset{g}{\,} X). \tag{94}
$$

Eq.(94) will be called *metric dual identity.*

2.8.4 Pseudo-Euclidean Metric Extensors on V

The metric extensor η Consider the vector spaces $\mathbf{V}$ and V ($\dim \mathbf{V} = \dim V = n$) and the dual bases $\{\mathbf{b}_\mu\}$ and $\{\boldsymbol{\beta}^\mu\}$ for $\mathbf{V}$ and V, $\boldsymbol{\beta}^\mu(\mathbf{b}_\nu) = \delta^\mu_\nu$. Moreover, denote the reciprocal basis of the basis $\{\boldsymbol{\beta}^\mu\}$ by $\{\beta_\mu\}$, with $\boldsymbol{\beta}^\mu \cdot \beta_\nu = \delta^\mu_\nu$, and $\mu, \nu = 0, 1, 2..., n$.

The extensor on V, say $\eta \in (1,1)$-*extV* defined for any $a \in \bigwedge^1 V$ by $a \mapsto \eta(a) \in \bigwedge^1 V$, such that

$$
\eta(a) = \boldsymbol{\beta}^0 a \boldsymbol{\beta}^0, \tag{95}
$$

is a well defined metric extensor on V (i.e., η is symmetric and non degenerated) and is called a *pseudo-Euclidean metric extensor for* V.

The main properties of η are:

(i) $\boldsymbol{\beta}^0$ is an eigenvector with eigenvalue 1, i.e., $\eta(\boldsymbol{\beta}^0) = \boldsymbol{\beta}^0$, and $\boldsymbol{\beta}^k$ ($k = 1, 2, ..., n$) are eigenvectors with eigenvalues -1, i.e., $\eta(\boldsymbol{\beta}^k) = -\boldsymbol{\beta}^k$.

(ii) The 1-forms of the canonical basis $\{\boldsymbol{\beta}^\mu\}$ are η orthonormal, i.e.,

$$
\boldsymbol{\beta}^\mu \underset{\eta}{\cdot} \boldsymbol{\beta}^\nu := \eta(\boldsymbol{\beta}^\mu) \cdot \boldsymbol{\beta}^\nu = \eta^{\mu\nu}, \tag{96}
$$

where the matrix with entries $(\eta^{\mu\nu})$ is the diagonal matrix $(1, -1, ..., -1)$.

Then, η has signature[28] $(1, n)$.

(iii) $\operatorname{tr}[\eta] = 2 - n$ and $\det[\eta] = (-1)^{n-1}$.

(iv) The extended of η has the same generator of η, i.e., for any $X \in \bigwedge U$, $\underline{\eta}(X) = \boldsymbol{\beta}^0 X \boldsymbol{\beta}^0$.

(v) η is orthogonal canonical, i.e., since

$$
\eta(\boldsymbol{\beta}^\mu) \cdot \boldsymbol{\beta}^\nu = \boldsymbol{\beta}^\mu \cdot \eta(\boldsymbol{\beta}^\nu),
$$

[28] Or, some as mathematicians say, signature $(1 - n)$.

it is $\eta = \eta^{\diamond}$ (recall that $\eta^{\diamond} = (\eta^{\dagger})^{-1} = (\eta^{-1})^{\dagger}$).Then, $\eta^2 = \mathrm{id}$.

Remark 2.1 When $\dim \mathbf{V} = 4$ and the signature of η is $(1,3)$ the pair $(\mathbf{V},\eta)$ is called a *Minkowski vector space* and η is called a Minkowski metric.

Metric Extensor $\boldsymbol{g}$ with the Same Signature of η Let $\boldsymbol{g}$ be a general metric extensor on V with the same signature as the η extensor defined by Eq.(95). Then, we have the following fundamental theorem[29], which will play a crucial role in the considerations that follows.

Theorem 2.1.For any pseudo-euclidean metric extensor $\boldsymbol{g}$ for V there exists a invertible $(1,1)$-extensor $\boldsymbol{h}$ (non unique!), such that

$$\boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}, \tag{97}$$

where η is defined by Eq.(95). The $(1,1)$-extensor field $\boldsymbol{h}$ is given by:

$$\boldsymbol{h}(a) = \sum_{\mu=0}^{n}\sqrt{|\lambda^{\mu}|}(a \cdot v^{\mu})\boldsymbol{\beta}^{\mu}, \tag{98}$$

where the $\lambda^{\mu}$are the eigenvalues of $\boldsymbol{g}$ and $v^{\mu} \in \bigwedge^1 V$ are the associated eigenvectors. The 1-form $\{v^{\mu}\}$ defines an *euclidean orthonormal basis* for V, i.e., $v^{\mu} \cdot v^{\nu} = \delta^{\mu\nu}$.

Proof We must prove that $\boldsymbol{h}$ in Eq.(98) satisfy the equation for composition of extensors $\boldsymbol{h}^{\dagger}\eta\boldsymbol{h} = \boldsymbol{g}$.

First we need to calculate the adjoint of $\boldsymbol{h}$. Take two arbitrary elements a, $b \in \bigwedge^1 V$. Then using Eq.(44) and Eq.(98) we can write

$$\begin{aligned} \boldsymbol{h}^{\dagger}(a) \cdot b &= a \cdot \boldsymbol{h}(b) \\ &= a \cdot (\sum_{\mu=0}^{n}\sqrt{|\lambda^{\mu}|}(b \cdot v^{\mu})\boldsymbol{\beta}^{\mu}) \\ &= (\sum_{\mu=0}^{n}\sqrt{|\lambda^{\mu}|}(a \cdot \boldsymbol{\beta}^{\mu})v^{\mu}) \cdot b, \end{aligned} \tag{99}$$

and from the non degeneracy of the scalar product we have

$$\boldsymbol{h}^{\dagger}(a) = \sum_{\mu=0}^{n}\sqrt{|\lambda^{\mu}|}(a \cdot \boldsymbol{\beta}^{\mu})v^{\mu}. \tag{100}$$

[29] Originally proved in [30].

Now, using Eqs.(98) and (100) we get

$$\begin{aligned}\boldsymbol{h}^{\dagger}\eta\boldsymbol{h}(a) &= \sum_{\mu=0}^{n}\sum_{\nu=0}^{n}\sqrt{|\lambda^{\mu}\lambda^{\nu}|}\eta(\boldsymbol{\beta}^{\mu})\cdot\boldsymbol{\beta}^{\nu}(a\cdot v^{\mu})v^{\nu} \\ &= \sqrt{|\lambda^{0}|^{2}}\eta(\boldsymbol{\beta}^{0})\cdot\boldsymbol{\beta}^{0}(a\cdot v^{0})v^{0} + \sum_{j=1}^{n}\sqrt{|\lambda^{j}\lambda^{0}|}\eta(\boldsymbol{\beta}^{j})\cdot\boldsymbol{\beta}^{0}(a\cdot v^{j})v^{0} \\ &+ \sum_{k=1}^{n}\sqrt{|\lambda^{0}\lambda^{k}|}\eta(\boldsymbol{\beta}^{0})\cdot\boldsymbol{\beta}^{k}(a\cdot v^{0})v^{k} + \sum_{j=1}^{n}\sum_{k=1}^{n}\sqrt{|\lambda^{j}\lambda^{k}|}\eta(\boldsymbol{\beta}^{j})\cdot\boldsymbol{\beta}^{k}(a\cdot v^{j})v^{k},\end{aligned}$$

and since $\eta(\boldsymbol{\beta}^{0}) = \boldsymbol{\beta}^{0}$ and $\eta(\boldsymbol{\beta}^{k}) = -\boldsymbol{\beta}^{k}$ $(k = 1, 2, ..., n)$, and $\boldsymbol{\beta}^{\mu}\cdot\boldsymbol{\beta}^{\nu} = \delta^{\mu\nu}$ we get

$$\begin{aligned}\boldsymbol{h}^{\dagger}\eta\boldsymbol{h}(a) &= \left|\lambda^{0}\right|(a\cdot v^{0})v^{0} - \sum_{j=1}^{n}\sum_{k=1}^{n}\sqrt{|\lambda^{j}\lambda^{k}|}\delta^{jk}(a\cdot v^{j})v^{k} \\ \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}(a) &= \left|\lambda^{0}\right|(a\cdot v^{0})v^{0} - \sum_{j=1}^{n}\left|\lambda^{j}\right|(a\cdot v^{j})v^{j}. \qquad (101)\end{aligned}$$

Finally using Eqs.(100) and (99) in Eq.(101), we have

$$\begin{aligned}\boldsymbol{h}^{\dagger}\eta\boldsymbol{h}(a) &= (a\cdot v^{0})\boldsymbol{g}(v^{0}) + \sum_{j=1}^{n}(a\cdot v^{j})\boldsymbol{g}(v^{j}) \\ &= \sum_{\mu=0}^{n}(a\cdot v^{\mu})\boldsymbol{g}(v^{\mu}) = \boldsymbol{g}(\sum_{\mu=0}^{n}(a\cdot v^{\mu})v^{\mu}) \\ \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}(a) &= \boldsymbol{g}(a).\ \blacksquare\end{aligned}$$

The $(1,1)$-extensor $\boldsymbol{h}$ given by Eq.(98) is invertible since

$$\begin{aligned}\boldsymbol{h}^{\dagger}\eta\boldsymbol{h} = \boldsymbol{g} &\Rightarrow \det[\boldsymbol{h}^{\dagger}]\det[\eta]\det[\boldsymbol{h}] = \det[\boldsymbol{g}] \\ &\Rightarrow (\det[\boldsymbol{h}])^{2}(-1) = \det[\boldsymbol{g}], \qquad (102)\end{aligned}$$

and this implies that $\det[\boldsymbol{h}] \neq 0$, i.e., $\boldsymbol{h}$ is non degenerated and thus invertible.

On the other hand the $(1,1)$-extensor $\boldsymbol{h}$ which satisfies Eq.(97) is certainly not unique. There is a *gauge freedom* which is represented by a local pseudo orthogonal transformation. Indeed, consider the $(1,1)$-extensor $\boldsymbol{h}' \equiv \boldsymbol{l}\boldsymbol{h}$, where $\boldsymbol{l}$ is an η-orthogonal $(1,1)$-extensor $\boldsymbol{l} = \boldsymbol{l}^{(\eta)}$, (or equivalently, $\boldsymbol{l}^{\dagger}\eta\boldsymbol{l} = \eta$) [30] and $\boldsymbol{h}$ is any $(1,1)$-extensor that satisfies Eq.(97) (e.g., the $\boldsymbol{h}$ given by Eq.(98)), then

$$(\boldsymbol{h}')^{\dagger}\eta\boldsymbol{h}' = (\boldsymbol{l}\boldsymbol{h})^{\dagger}\eta\boldsymbol{l}\boldsymbol{h} = \boldsymbol{h}^{\dagger}l^{\dagger}\eta\boldsymbol{l}\boldsymbol{h} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h} = \boldsymbol{g}.$$

i.e., $\boldsymbol{h}'$ also satisfies Eq.(97).

Remark 2.2 When $\dim V = 4$ and the signature of $\boldsymbol{g}$ (and, of course, of η) is $(1,3)$ $\boldsymbol{g}$ is said a *Lorentz metric extensor.*

[30] I.e., a *transformation* $l(a) \underset{\eta}{\cdot} l(b) = a \underset{\eta}{\cdot} b$.

2.8.5 Some Remarkable Results

We recall now some remarkable results that will be need in repeatedly in the following sections.

Golden Rule Let $\boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}$ be defined as in Eq.(97). Then for any $X, Y \in \bigwedge U$ it holds the following remarkable identity known as the *golden rule* [62]:

$$\underline{\boldsymbol{h}}(X \underset{\boldsymbol{g}}{*} Y) = \underline{\boldsymbol{h}}(X) \underset{\eta}{*} \underline{\boldsymbol{h}}(Y), \tag{103}$$

where $*$ denotes here as previously agreed any one of the multiform products.

Hodge Star Operators The canonical Hodge star operator in $\bigwedge V$ is defined first by its action on $\bigwedge^p V$ by

$$\begin{gathered} \star : \bigwedge^p V \to \bigwedge^{n-p} V, \\ X \mapsto \star X := \tilde{X} \lrcorner \tau = \tilde{X}\tau, \end{gathered} \tag{104}$$

where τ is the canonical volume element[31] $\tau = \boldsymbol{\beta}^0 \wedge \boldsymbol{\beta}^1 \wedge \boldsymbol{\beta}^2 \wedge ... \wedge \boldsymbol{\beta}^n$ and next extended by linearity to all $\bigwedge V$.

Then the Hodge star operators associated to the metric extensors η and $\boldsymbol{g}$ just introduced above are given by:

$$\begin{array}{ll} \underset{\eta}{\star} : \bigwedge^p V \to \bigwedge^{n-p} V, & \underset{\boldsymbol{g}}{\star} : \bigwedge^p V \to \bigwedge^{n-p} V, \\ \underset{\eta}{\star} X := \tilde{X} \lrcorner \tau_\eta = \tilde{X}\tau_\eta, & \underset{\boldsymbol{g}}{\star} X := \tilde{X} \lrcorner \tau_{\boldsymbol{g}} = \tilde{X}\tau_{\boldsymbol{g}}, \end{array} \tag{105}$$

where

$$\tau_\eta = \sqrt{|\det \eta|}\tau, \; \tau_{\boldsymbol{g}} = \sqrt{|\det \boldsymbol{g}|}\tau. \tag{106}$$

Relation Between the Hodge Star Operators of $\boldsymbol{g}$ and the Canonical Hodge Star Operator We then have the following nontrivial results relating $\underset{\boldsymbol{g}}{\star}$ and $\star$. For any $X \in \bigwedge V$,

$$\underset{\boldsymbol{g}}{\star} X = \frac{1}{\sqrt{|\det \boldsymbol{g}|}} \underline{\boldsymbol{g}}(\tilde{X} \lrcorner \tau) = \frac{1}{\sqrt{|\det \boldsymbol{g}|}} \underline{\boldsymbol{g}} \circ \star X, \tag{107}$$

[31] Note that if the basis $\{\boldsymbol{\beta}^\mu\}$ is (canonical) orthonormal we can write the volume element as $\tau = \boldsymbol{\beta}^0\boldsymbol{\beta}^1\boldsymbol{\beta}^2...\boldsymbol{\beta}^n$.

Relation Between the Hodge Star Operators of $\boldsymbol{g}$ and η Putting

$$\boldsymbol{h}^{\clubsuit} := \boldsymbol{h}^{\dagger -1}, \tag{108}$$

the following result is valid when $\boldsymbol{g}$ and η has the same signature

$$\underset{g}{\star} = \operatorname{sgn}(\det \boldsymbol{h}) \underline{\boldsymbol{h}}^{\dagger} \circ \underset{\eta}{\star} \circ \underline{\boldsymbol{h}}^{\clubsuit}. \tag{109}$$

Moreover if we suppose that $\boldsymbol{h}$ is continuously connected to the identity extensor which has determinant equal to 1 we can write

$$\underset{g}{\star} = \underline{\boldsymbol{h}}^{\dagger} \underset{\eta}{\star} \underline{\boldsymbol{h}}^{\clubsuit} \tag{110}$$

Useful Identities We shall list some identities involving contractions, exterior product and the Hodge star operator that will be need for some derivations in Sections 5, 6,7 and Appendices D and E. These are

For any metric g and for any $A_r \in \bigwedge^r V$ and $B_s \in \bigwedge^s V$, $r, s \geq 0$:

$$\begin{aligned}
&A_r \wedge \underset{g}{\star} B_s = B_s \wedge \underset{g}{\star} A_r; \quad r = s\\
&A_r \underset{g^{-1}}{\cdot} \underset{g}{\star} B_s = B_s \cdot \underset{g}{\star} A_r; \quad r + s = n\\
&A_r \wedge \underset{g}{\star} B_s = (-1)^{r(s-1)} \underset{g}{\star} (\tilde{A}_r \underset{g^{-1}}{\lrcorner} B_s); \quad r \leq s\\
&A_r \lrcorner \underset{g}{\star} B_s = (-1)^{rs} \underset{g}{\star} (\tilde{A}_r \wedge B_s); \quad r + s \leq n\\
&\underset{g}{\star} A_r = \tilde{A}_r \underset{g^{-1}}{\lrcorner} \tau_{\underset{g}{\circ}} = \tilde{A}_r \, \tau_g\\
&\underset{g}{\star} \tau_g = \operatorname{sgn} \boldsymbol{g}; \quad \underset{g}{\star} 1 = \tau_g.
\end{aligned} \tag{111}$$

Finally, we shall need also the following result. Let $\{\varepsilon^{\mu}\}$ be an arbitrary basis of U such that $\varepsilon^{\mu} \underset{g^{-1}}{\cdot} \varepsilon^{\mu} = g^{\mu\nu}$. Then, if

$$\varepsilon^{\mu_1 \ldots \mu_p} = \varepsilon^{\mu_1} \wedge \ldots \wedge \varepsilon^{\mu_p}, \varepsilon^{\nu_{p+1} \ldots \nu_n} = \varepsilon^{\nu_{p+1}} \wedge \ldots \wedge \varepsilon^{\nu_n}$$

we have, with $g^{\mu\nu} g_{\mu\alpha} = \delta^{\nu}_{\alpha}$,

$$\underset{g}{\star} \varepsilon^{\mu_1 \ldots \mu_p} = \frac{1}{(n-p)!} \sqrt{|\det(g_{\mu\nu})|} g^{\mu_1 \nu_1} \ldots g^{\mu_p \nu_p} \varepsilon_{\nu_1 \ldots \nu_n} \varepsilon^{\nu_{p+1} \ldots \nu_n}, \tag{112}$$

where $\det(g_{\mu\nu})$ is the classical determinant of the matrix with entries $g_{\mu\nu}$.

3 Multiform Functions and Multiform Functionals

3.1 Multiform Functions of Real Variable

A mapping

$$X : S \to \bigwedge V, \ (S \subseteq \mathbb{R}), \tag{113}$$

is called a *multiform function of real variable.*

For simplicity reasons when the image of X is a scalar, a 1-form, a biform, etc..., X is said a scalar function, 1-form function, biform function, etc...

3.1.1 Limit and Continuity

The concepts of limit and continuity may be easily introduced following a path analog to the one used in the theory of ordinary functions.

Then, a multiform $L \in \bigwedge V$ is said to be the limit of $X(\lambda)$ *for* $\lambda \in S$ approaching $\lambda_0 \in S$ iff for any real number $\varepsilon > 0$ there exists as real number $\delta > 0$ such that[32] $\|X(\lambda) - L\| < \varepsilon$, if $0 < |\lambda - \lambda_0| < \delta$. As usual, such a concept will be denoted by $\lim\limits_{\lambda \to \lambda_0} X(\lambda) = L$.

The theorems for the limits of multiform functions are completely analogous to the ones of the theory of ordinary functions, i.e.,

$$\lim_{\lambda \to \lambda_0} X(\lambda) + Y(\lambda) = \lim_{\lambda \to \lambda_0} X(\lambda) + \lim_{\lambda \to \lambda_0} Y(\lambda). \tag{114}$$

$$\lim_{\lambda \to \lambda_0} X(\lambda) * Y(\lambda) = \lim_{\lambda \to \lambda_0} X(\lambda) * \lim_{\lambda \to \lambda_0} Y(\lambda), \tag{115}$$

where $*$ denotes any one of the multiform products, i.e., $(\wedge)$, $(\cdot)$, $(\lrcorner, \llcorner)$ or (*Clifford product*).

$X(\lambda)$ is said to be *continuous* at $\lambda_0 \in S$ iff $\lim\limits_{\lambda \to \lambda_0} X(\lambda) = X(\lambda_0)$ and sum and product of continuous multiform functions are continuous.

3.1.2 Derivative

The notion of derivative of a multiform function of a real variable is also formulated in completely analogy to the one used in the theory of ordinary functions. Then the derivative of $X(\lambda)$ *at* λ_0 is defined by

$$X'(\lambda_0) := \lim_{\lambda \to \lambda_0} \frac{X(\lambda) - X(\lambda_0)}{\lambda - \lambda_0}. \tag{116}$$

The rules for derivations, as expected, are completely analogous to the ones valid for ordinary functions.

$$(X + Y)' = X' + Y', \tag{117}$$

$$(X * Y)' = X' * Y + X * Y' \text{ (Leibniz rule)}, \tag{118}$$

$$(X \circ \phi)' = (X' \circ \phi)\phi' \text{ (chain rule)}, \tag{119}$$

where $*$, as above means $(\wedge)$, $(\cdot)$, $(\lrcorner, \llcorner)$ or (*Clifford product*) and ϕ is a real ordinary function (ϕ' is the derivative of ϕ).

In the space of derivable multiform functions of real variable, we introduce the symbol $\dfrac{d}{d\lambda}$ (derivative operator), by $\dfrac{d}{d\lambda}X(\lambda) = X'(\lambda)$.

We can easily generalize all the above rules for multiform functions of several real variables, so it is not necessary any additional comments. Now we will introduce the real important objects need in this paper.

[32] For any $A \in \bigwedge V$, $\|X\| := \sqrt{\langle X \cdot X \rangle_0}$

3.2 Multiform Functions of Multiform Variables

A mapping $F : \bigwedge^{\diamond} V \to \bigwedge V$ is called a *multiform function of a multiform variable* and when $F(X)$ is a scalar, a 1-form, a biform, etc..., then F is said to be a scalar function*, a* 1*-form function, a biform function, etc...*

3.2.1 Limit and Continuity

Let $F : \bigwedge^{\diamond} V \to \bigwedge V$. A *multiform* $M \in \bigwedge V$ is said to be the limit of $F(X)$ *for* $X \in \bigwedge^{\diamond} V$ *approaching* $X_0 \in \bigwedge V$ iff for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that $\|F(X) - M\| < \varepsilon$, if $0 < \|X - X_0\| < \delta$. This will be denoted by $\lim\limits_{X \to X_0} F(X) = M$.

When we are considering scalar functions of multiform variables $\bigwedge^{\diamond} V \ni X \mapsto \Phi(X) \in \mathbb{R}$, the definition $\lim\limits_{X \to X_0} \Phi(X) = \alpha$ is simply the statement: for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that $|\Phi(X) - \alpha| < \varepsilon$, if $0 < \|X - X_0\| < \delta$.

The limit theorems are analogous to the ones of the theory of ordinary functions, i.e.,

$$\lim_{X \to X_0} F(X) + G(X) = \lim_{X \to X_0} F(X) + \lim_{X \to X_0} G(X). \tag{120}$$

$$\lim_{X \to X_0} F(X) * G(X) = \lim_{X \to X_0} F(X) * \lim_{X \to X_0} G(X), \tag{121}$$

where $*$ means as now usual $(\wedge)$, $(\cdot)$, $(\lrcorner, \llcorner)$ or the (Clifford product).

Let $F : \bigwedge^{\diamond} V \to \bigwedge V$ be a multiform function of multiform variable. We say that F is continuous at $X_0 \in \bigwedge V$ iff $\lim\limits_{X \to X_0} F(X) = F(X_0)$.

The sum $X \mapsto (F + G)(X) = F(X) + G(X)$ and any product of continuous multiform functions of multiform variable $X \mapsto (F * G)(X) = F(X) * G(X)$ are also continuous multiform functions of multiform variable.

3.2.2 Differentiability

A multiform function of multiform variable $F : \bigwedge^{\diamond} V \to \bigwedge V$ is said differentiable at $X_0 \in \bigwedge V$ iff it exists $f_{X_0} \in extV$, such that

$$\lim_{X \to X_0} \frac{F(X) - F(X_0) - f_{X_0}(X - X_0)}{\|X - X_0\|} = 0. \tag{122}$$

If such f_{X_0} *exists*, it must be unique and will be called the *differential* of F *at* X_0.

3.2.3 The Directional Derivative $A \cdot \partial_X$

Let $F : \bigwedge^{\diamond} V \to \bigwedge V$ be differentiable at X. Take a real number $\lambda \neq 0$ and an arbitrary multiform A. Denote also by $\langle\rangle_X : \bigwedge V \to \bigwedge^{\diamond} V$ Then the limit

$$\lim_{\lambda \to 0} \frac{F(X + \lambda \langle A \rangle_X) - F(X)}{\lambda} \tag{123}$$

exists. It will be denoted by $F'_A(X)$ (or, $A \cdot \partial_X F(X)$)[33] and called the directional derivative of F *at* X *in the direction of the multiform* A. The operator $A \cdot \partial_X$ is said to be the directional derivative operator in the direction of A. We have:

$$F'_A(X) = A \cdot \partial_X F(X) := \lim_{\lambda \to 0} \frac{F(X + \lambda \langle A \rangle_X) - F(X)}{\lambda}, \tag{124}$$

or in an equivalent form

$$F'_A(X) = A \cdot \partial_X F(X) = \left. \frac{d}{d\lambda} F(X + \lambda \langle A \rangle_X) \right|_{\lambda=0}. \tag{2.10}$$

The directional derivative of a differentiable multiform function $F : \bigwedge^{\diamond} V \to \bigwedge V$ is linear with respect to the direction multiform, i.e., for all $\alpha, \beta \in \mathbb{R}$ and $A, B \in \bigwedge V$

$$F'_{\alpha A + \beta B}(X) = \alpha F'_A(X) + \beta F'_B(X), \tag{125}$$

or yet

$$(\alpha A + \beta B) \cdot \partial_X F(X) = \alpha A \cdot \partial_X F(X) + \beta A \cdot \partial_X F(X). \tag{126}$$

We give now the main properties of the directional derivative.

Let $F, G : \bigwedge^{\diamond} V \to \bigwedge V$ be differentiable, the sum $X \mapsto (F + G)(X) = F(X) + G(X)$ and the products $X \mapsto (F * G)(X) = F(X) * G(X)$, where $*$ means $(\wedge)$, $(\cdot)$, $(\lrcorner, \llcorner)$ or (*Clifford product*), are also differentiable and we have

$$(F + G)'_A(X) = F'_A(X) + G'_A(X), \tag{127}$$

$$(F * G)'_A(X) = F'_A(X) * G(X) + F(X) * G'_A(X) \text{ (Leibniz rule)}. \tag{128}$$

3.2.4 Chain Rules

Let $F, G : \bigwedge^{\diamond} V \to \bigwedge V$ be differentiable. The composition $X \mapsto (F \circ G)(X) = F(\langle G(X) \rangle_Y)$, with $Y \in \mathrm{dom} F$ is differentiable and

$$(F \circ G)'_A(X) = F'_{G'_A(X)}(\langle G(X) \rangle_Y). \tag{129}$$

[33]For some few special cases we use yet some special symbols.

Let $X \mapsto F(X)$ and $\lambda \mapsto X(\lambda)$ be differentiable multiform functions, the first one of multiform variable and the second of real variable. The composition $\lambda \mapsto (F \circ X)(\lambda) = F(\langle X(\lambda)\rangle_Y)$, with $Y \in \text{dom}F$ is a differentiable multiform function of real variable and

$$(F \circ X)'(\lambda) = F'_{X'(\lambda)}(\langle X(\lambda)\rangle_Y). \tag{130}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \bigwedge^{\diamond} V \to \mathbb{R}$ respectively an ordinary differentiable function and a scalar differentiable multiform function of multiform variable. The composition $\phi \circ \Psi : \bigwedge^{\diamond} V \to \mathbb{R}$ such that $(\phi \circ \Psi)(X) = \phi(\Psi(X))$ is differentiable multiform function of multiform variable and

$$(\phi \circ \Psi)'_A(X) = \phi'(\Psi(X))\Psi'_A(X). \tag{131}$$

In resume we have:

$$A \cdot \partial_X (F + G)(X) = A \cdot \partial_X F(X) + A \cdot \partial_X G(X). \tag{132}$$

$$A \cdot \partial_X (F * G)(X) = A \cdot \partial_X F(X) * G(X) + F(X) * A \cdot \partial_X G(X). \tag{133}$$

$$A \cdot \partial_X (F \circ G)(X) = A \cdot \partial_X G(X) \cdot \partial_Y F(\langle G(X)\rangle_Y), \text{ with } Y \in domF. \tag{134}$$

$$\frac{d}{d\lambda}(F \circ X)(\lambda) = \frac{d}{d\lambda}X(\lambda) \cdot \partial_Y F(\langle X(\lambda)\rangle_Y), \text{ with } Y \in domF. \tag{135}$$

$$A \cdot \partial_X (\phi \circ \Psi)(X) = \frac{d}{d\mu}\phi(\Psi(X)) A \cdot \partial_X \Psi(X). \tag{136}$$

3.3 The Derivative Mapping ∂_X

Let $F : \bigwedge^{\diamond} V \to \bigwedge V$ be differentiable at X. Take any pair of arbitrary reciprocal basis for V, say $(\{\varepsilon^j\}, \{\varepsilon_j\})$, $(\varepsilon^j \cdot \varepsilon_i = \delta_i^j)$. Then, it exists a well defined multiform function (i.e., it is independent of the pair $(\{\varepsilon^j\}, \{\varepsilon_j\})$ depending only on F) called the derivative of F *at* X and given by

$$F'(X) = \partial_X F(X) := \sum_J \frac{1}{\nu(J)!} \varepsilon^J F'_{\varepsilon_J}(X) = \sum_J \frac{1}{\nu(J)!} \varepsilon_J F'_{\varepsilon^J}(X). \tag{137}$$

The main properties of $\partial_X F$ are:

(i) Let $X \mapsto \Phi(X)$ be a scalar multiform function, then

$$A \cdot \partial_X \Phi(X) = A \cdot (\partial_X \Phi(X)). \tag{138}$$

(ii) Let $X \mapsto F(X)$, $X \mapsto G(X)$ and $X \mapsto \Phi(X)$ be differentiable multiform functions. Then

$$\partial_X (F + G)(X) = \partial_X F(X) + \partial_X G(X). \tag{139}$$

$$\partial_X (\Phi G)(X) = \partial_X \Phi(X) G(X) + \Phi(X) \partial_X G(X) \text{ (Leibniz rule)}. \tag{140}$$

3.4 Examples

We present now some examples which illustrate some of the most important formulas of the calculus of multiform functions of multiform variables and which will be used in the following sections.

Example 3.1 Let $\bigwedge^{\diamond} V \ni X \mapsto X \cdot X \in \mathbb{R}$. Let us calculate $A \cdot \partial_X(X \cdot X)$ and $\partial_X(X \cdot X)$.

$$
\begin{aligned}
A \cdot \partial_X(X \cdot X) &= \frac{d}{d\lambda}(X + \lambda \langle A\rangle_X) \cdot (X + \lambda \langle A\rangle_X)\Big|_{\lambda=0} \\
&= \frac{d}{d\lambda}(X \cdot X + 2\lambda \langle A\rangle_X \cdot X + \lambda^2 \langle A\rangle_X \cdot \langle A\rangle_X)\Big|_{\lambda=0},
\end{aligned}
$$

$$
A \cdot \partial_X(X \cdot X) = 2\langle A\rangle_X \cdot X = 2A \cdot X. \tag{141a}
$$

$$
\partial_X(X \cdot X) = \sum_J \frac{1}{\nu(J)!}\varepsilon^J \varepsilon_J \cdot \partial_X(X \cdot X) = \sum_J \frac{1}{\nu(J)!}\varepsilon^J 2(\varepsilon_J \cdot X) = 2X. \tag{141b}
$$

Example 3.2 Let $\bigwedge^{\diamond} V \ni X \mapsto B \cdot X \in \mathbb{R}$, with $B \in \bigwedge V$. Let us calculate $A \cdot \partial_X(B \cdot X)$ and $\partial_X(B \cdot X)$.

$$
A \cdot \partial_X(B \cdot X) = \frac{d}{d\lambda} B \cdot (X + \lambda \langle A\rangle_X)\Big|_{\lambda=0} = B \cdot \langle A\rangle_X = A \cdot \langle B\rangle_X, \tag{142a}
$$

$$
\partial_X(B \cdot X) = \sum_J \frac{1}{\nu(J)!}\varepsilon^J \varepsilon_J \cdot \partial_X(B \cdot X) = \sum_J \frac{1}{\nu(J)!}\varepsilon^J(\varepsilon_J \cdot \langle B\rangle_X) = \langle B\rangle_X. \tag{142b}
$$

Example 3.3 Let $\bigwedge^{\diamond} V \ni X \mapsto (BXC) \cdot X \in \mathbb{R}$, with $B, C \in \bigwedge V$. Let us calculate:
$A \cdot \partial_X((BXC) \cdot X)$ and $\partial_X((BXC) \cdot X)$.

$$
\begin{aligned}
A \cdot \partial_X((BXC) \cdot X) &= \frac{d}{d\lambda}(B(X + \lambda \langle A\rangle_X)C) \cdot (X + \lambda \langle A\rangle_X)\Big|_{\lambda=0} \\
&= \frac{d}{d\lambda}((BXC) \cdot X + \lambda(B\langle A\rangle_X C) \cdot X + \lambda(BXC) \cdot \langle A\rangle_X \\
&\qquad + \lambda^2(B\langle A\rangle_X C) \cdot \langle A\rangle_X)\big|_{\lambda=0} \\
&= (B\langle A\rangle_X C) \cdot X + (BXC) \cdot \langle A\rangle_X = \langle A\rangle_X \cdot (BXC + \widetilde{B}X\widetilde{C}),
\end{aligned} \tag{143a}
$$

$$
A \cdot \partial_X((BXC) \cdot X) = A \cdot \left\langle BXC + \widetilde{B}X\widetilde{C}\right\rangle_X.
$$

$$
\begin{aligned}
\partial_X((BXC) \cdot X) &= \sum_J \frac{1}{\nu(J)!}\varepsilon^J \varepsilon_J \cdot \partial_X(BXC \cdot X) \\
&= \sum_J \frac{1}{\nu(J)!}\varepsilon^J(\varepsilon_J \cdot \left\langle BXC + \widetilde{B}X\widetilde{C}\right\rangle_X),
\end{aligned}
$$

$$
\partial_X((BXC) \cdot X) = \left\langle BXC + \widetilde{B}X\widetilde{C}\right\rangle_X. \tag{143b}
$$

In this example we essentially utilized the nontrivial multiform identity $(AXB)\cdot Y = X\cdot(\widetilde{A}Y\widetilde{B})$.

Example 3.4 Consider $x\wedge B$, with $x\in\bigwedge^1 V$ and $B\in\bigwedge^2 V$. Then $x\wedge B\in\bigwedge^3 V$. Let us calculate the directional derivative $a\cdot\partial_x(x\wedge B)$, the *divergent* $\partial_x\lrcorner(x\wedge B)$, the *rotational* $\partial_x\wedge(x\wedge B)$ and the gradient $\partial_x(x\wedge B)$.

$$a\cdot\partial_x(x\wedge B) = \frac{d}{d\lambda}(x+\lambda a)\wedge B\bigg|_{\lambda=0} = a\wedge B, \tag{144a}$$

$$\partial_x\lrcorner(x\wedge B) = \sum_{j=1}^n \varepsilon^j\lrcorner\varepsilon_j\cdot\partial_x(x\wedge B) = \sum_{j=1}^n \varepsilon^j\lrcorner(\varepsilon_j\wedge B) = (n-2)B. \tag{144b}$$

$$\partial_x\wedge(x\wedge B) = \sum_{j=1}^n \varepsilon^j\wedge\varepsilon_j\cdot\partial_x(x\wedge B) = \sum_{j=1}^n \varepsilon^j\wedge(\varepsilon_j\wedge B) = 0. \tag{144c}$$

$$\begin{aligned}\partial_x(x\wedge B) &= \sum_{j=1}^n \varepsilon^j\varepsilon_j\cdot\partial_x(x\wedge B) = \sum_{j=1}^n \varepsilon^j(\varepsilon_j\wedge B)\\ &= \sum_{j=1}^n (\varepsilon^j\lrcorner(\varepsilon_j\wedge B) + \varepsilon^j\wedge(\varepsilon_j\wedge B)),\end{aligned} \tag{144d}$$

$$\partial_x(x\wedge B) = (n-2)B, \qquad (n=\dim V). \tag{144e}$$

Example 3.5 Consider $x\lrcorner\, B\in\bigwedge^1 V$. Let us calculate $a\cdot\partial_x(x\lrcorner B)$, $\partial_x\lrcorner(x\lrcorner B)$, $\partial_x\wedge(x\lrcorner B)$ and $\partial_x(x\lrcorner B)$

$$a\cdot\partial_x(x\lrcorner B) = \frac{d}{d\lambda}(x+\lambda a)\lrcorner B\bigg|_{\lambda=0} = a\lrcorner B. \tag{145a}$$

$$\partial_x\lrcorner(x\lrcorner B) = \sum_{j=1}^n \varepsilon^j\lrcorner\varepsilon_j\cdot\partial_x(x\lrcorner B) = \sum_{j=1}^n \varepsilon^j\lrcorner(\varepsilon_j\lrcorner B) = \sum_{j=1}^n (\varepsilon^j\wedge\varepsilon_j)\lrcorner\, B = 0, \tag{145b}$$

$$\partial_x\wedge(x\lrcorner B) = \sum_{j=1}^n \varepsilon^j\wedge\varepsilon_j\cdot\partial_x(x\lrcorner B) = \sum_{j=1}^n \varepsilon^j\wedge(\varepsilon_j\lrcorner B) = 2B. \tag{145c}$$

$$\partial_x(x\lrcorner B) = \sum_{j=1}^n \varepsilon^j\varepsilon_j\cdot\partial_x(x\lrcorner B) = \sum_{j=1}^n \varepsilon^j(\varepsilon_j\lrcorner B) \tag{145d}$$

$$\partial_x(x\lrcorner B) = \sum_{j=1}^n (\varepsilon^j\lrcorner(\varepsilon_j\lrcorner B) + \varepsilon^j\wedge(\varepsilon_j\lrcorner B)) = 2B. \qquad (n=\dim V) \tag{145e}$$

Example 3.6 Let $X\in\bigwedge^r V$ and consider the multiform function $F:\bigwedge^r\ni X\mapsto X\wedge\star X\in\bigwedge^n V$. Let us calculate the directional derivative $W\cdot\partial_X F$, with $W\in\bigwedge^r V$ and $\partial_X F$. We have

$$\begin{aligned}W\cdot\partial_X F(X) &= \lim_{\lambda\to 0}\frac{(X+\lambda W)\wedge\star(X+\lambda W) - X\wedge\star X}{\lambda}\\ &= 2W\wedge\star X.\end{aligned} \tag{146}$$

On the other hand using Eq.(137), we have:

$$
\begin{aligned}
\partial_X F(X) &= \sum (\frac{1}{r!}\varepsilon^{j_1} \wedge ... \wedge \varepsilon^{j_r})[(\varepsilon_{j_1} \wedge ... \wedge \varepsilon_{j_r}) \cdot \partial_X F(X)] \\
&= 2\sum \frac{1}{r!}\varepsilon^{j_1} \wedge ... \wedge \varepsilon^{j_r}[(\varepsilon_{j_1} \wedge ... \wedge \varepsilon_{j_r}) \wedge \star X] \\
&= 2\sum \frac{1}{r!}\varepsilon^{j_1} \wedge ... \wedge \varepsilon^{j_r}(-1)^{r(r-1)}[\star((\varepsilon_{j_1} \widetilde{\wedge ... \wedge} \varepsilon_{j_r}) \lrcorner X))] \\
&= 2\sum \frac{1}{r!}\varepsilon^{j_1} \wedge ... \wedge \varepsilon^{j_r}[(\varepsilon_{j_1} \wedge ... \wedge \varepsilon_{j_r}) \cdot X)\tau] \\
&= 2X\tau = 2(-1)^{\frac{r}{2}(r-1)}\tilde{X} \lrcorner \tau = 2(-1)^{\frac{r}{2}(r-1)} \star X.
\end{aligned}
$$

Remark 3.1 Sometimes the directional derivative of a multiform function $F : \bigwedge^r \ni X \mapsto X \wedge \star X \in \bigwedge^n$ in the direction of $W = \boldsymbol{\delta} X \in \bigwedge^r V$ is written as

$$
\boldsymbol{\delta} F := \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} \tag{147}
$$

and called the *variational derivative* of F and $\frac{\partial F}{\partial X}$ is called the algebraic derivative of F. Now, since $\boldsymbol{\delta} F = W \cdot \partial_X$ we see that we have the identification for our multiform function

$$
\boldsymbol{\delta} X \cdot \partial_X F = (-1)^{\frac{r}{2}(r-1)} \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X}. \tag{148}
$$

3.5 The Operators $\partial_X *$ and their t-distortions

We introduce now the linear differential operators $\partial_X *$ defined by

$$
\partial_X * F(X) := \sum_J \frac{1}{\nu(J)!}\varepsilon^J * \varepsilon_J \cdot \partial_X F(X), \tag{149}
$$

where $*$ denotes any one of the multiform products $(\wedge)$, $(\cdot)$, $(\lrcorner, \llcorner)$ or (*Clifford product*).

Those linear differentiable operators are well defined since the multiforms in the second member of Eq.(149) depends only on the multiform function F and do not depend on the pair of arbitrary reciprocal basis used.

Note that if $*$ refers to the Clifford product then the operator $\partial_X *$ is precisely the operator ∂_X (i.e., the derivative operator). Sometimes, ∂_X is called the *gradient operator* and $\partial_X F$ is said to be the *gradient of* F.

Also, the operator $\partial_X \wedge$ is called the *rotational operator* and $\partial_X \wedge F$ is said to be the *rotational of* F.

The operators $\partial_X \cdot$ and $\partial_X \lrcorner$ are called divergent operators, $\partial_X \cdot F$ is said to be the *scalar divergent and* $\partial_X \lrcorner F$ is said to be the *contracted divergent*.

Let $t \in (1,1)$-$extV$. We define the linear differential operators $\underline{t}(\partial_X)*$ by

$$
\underline{t}(\partial_X) * F(X) := \sum_J \frac{1}{\nu(J)!}\underline{t}(\varepsilon^J) * \varepsilon_J \cdot \partial_X F(X), \tag{150}
$$

where $*$ is any one of the multiform products $(\wedge)$, $(\cdot)$, $(\lrcorner,\llcorner)$ or (*Clifford product*).

Those linear differential operators are well defined since they depends only on F and do not depend on the pair of reciprocal basis $(\{\varepsilon^j\},\{\varepsilon_j\})$. We call $\underline{t}(\partial_X)*$ a t-distortion of $\partial_X *$. Special case have special names, i.e., we call $\underline{t}(\partial_X)\wedge F$ the t-rotational of F, $\underline{t}(\partial_X)\cdot F$ is the t-scalar divergent of F and $\underline{t}(\partial_X)\lrcorner F$ is *t-contracted divergent of* F. Finally, $\underline{t}(\partial_X)$ is the *t- gradient operator and* $\underline{t}(\partial_X)F$ is read as the *t-gradient of* F.

3.6 Multiform Functionals $\mathcal{F}_{(X^1,\ldots,X^k)}[t]$

Let

$$F:\underbrace{\bigwedge\nolimits^q V\times\cdots\times\bigwedge\nolimits^q V}_{k\text{ copies}}\to\bigwedge V,$$
$$(X^1,\ldots,X^k)\mapsto F(X^1,\ldots,X^k)=F_{(X^1,\ldots,X^k)}. \tag{151}$$

A mapping

$$\mathcal{F}_{(X^1,\ldots,X^k)}:(p,q)\text{-}extV\to\bigwedge V,$$
$$t\mapsto\mathcal{F}_{(X^1,\ldots,X^k)}[t]:=F[t(X^1),\ldots,t(X^k)], \tag{152}$$

will be called a *multiform functional (p,q)-extensor variable induced by the function* F. We say also that F is the *generator of* $\mathcal{F}_{(X^1,\ldots,X^k)}$.

When $\mathcal{F}[t]$ is scalar valued, 1-form valued, biform valued, etc..., $\mathcal{F}$ it is said to be a *scalar functional, 1-form functional, biform functional, etc..., of a (p,q)-extensor variable.*

3.7 Derivatives of Induced Multiform Functionals

Let $(X^1,\ldots,X^k)\mapsto F(X^1,\ldots,X^k)$ be a differentiable multiform function, i.e., differentiable with respect to each one of its q-multiform variables $(X^1,\ldots,X^k)$. Then, the partial derivatives of F, $\partial_{X^1}F,\ldots,\partial_{X^k}F$ exist. We can then construct exactly k multiform functionals of a (p,q)-extensor variable $t\in(p,q)\text{-}extV$ which are induced by them, namely

$$t\mapsto(\partial_{X^1}F)_{(X^1,\ldots,X^k)}[t]:=\partial_{t(X^1)}F[t(X^1),\ldots,t(X^k)],\ldots,$$
$$t\mapsto(\partial_{X^k}F)_{(X^1,\ldots,X^k)}[t]:=\partial_{t(X^k)}F[t(X^1),\ldots,t(X^k)]. \tag{153}$$

3.7.1 The A-Directional $A\cdot\partial_t$ Derivative of a Multiform Functional

Let $A\in\bigwedge V$ be an arbitrary multiform. The multiform functional of (p,q)-extensor variable t,

$$(p,q)\text{-}extV\ni t\mapsto A\cdot\partial_t\mathcal{F}_{(X^1,\ldots,X^k)}[t]\in\bigwedge V \tag{154}$$

such

$$
\begin{aligned}
A \cdot \partial_t \mathcal{F}_{(X^1,\dots,X^k)}[t] &:= A \cdot X^1 (\partial_{X^1} F)_{(X^1,\dots,X^k)}[t] + \cdots + A \cdot X^k (\partial_{X^k})_{(X^1,\dots,X^k)}[t] \\
&= \sum_{i=1}^{k} A \cdot X^i (\partial_{X^i} F)_{(X^1,\dots,X^k)}[t] \\
&= \sum_{i=1}^{k} A \cdot X^i \partial_{t(X^i)} F[t(X^1),\dots,t(X^k)] \qquad (155)
\end{aligned}
$$

is said to be [34] the A-*directional derivative of the functional* $\mathcal{F}_{(X^1,\dots,X^k)}[t]$. A convenient notation in order to write short formulas is $\mathcal{F}'_{(X^1,\dots,X^k)A}[t]$, i.e., we have

$$
\mathcal{F}'_{(X^1,\dots,X^k)A}[t] \equiv A \cdot \partial_t \mathcal{F}_{(X^1,\dots,X^k)}[t]. \qquad (156)
$$

Observe that $A \cdot \partial_t$ is linear with respect to the direction multiform A, i.e., for all $\alpha, \beta \in \mathbb{R}$ and $A, B \in \bigwedge V$,

$$
(\alpha A + \beta B) \cdot \partial_t = \alpha A \cdot \partial_t + \beta B \cdot \partial_t. \qquad (157)
$$

The operator $A \cdot \partial_t$ possess three important properties:

$$
A \cdot \partial_t \ (\mathcal{F}_{(X^1,\dots,X^k)} + \mathcal{G}_{(X^1,\dots,X^k)})[t] = A \cdot \partial_t \ \mathcal{F}_{(X^1,\dots,X^k)}[t] + A \cdot \partial_t \ \mathcal{G}_{(X^1,\dots,X^k)}[t]. \qquad (158)
$$

$$
A \cdot \partial_t \ (\alpha \mathcal{F}_{(X^1,\dots,X^k)})[t] = \alpha (A \cdot \partial_t \ \mathcal{F}_{(X^1,\dots,X^k)}[t]) \qquad (159)
$$

$$
A \cdot \partial_t (\mathcal{F}_{(X^1,\dots,X^k)} B)[t] = (A \cdot \partial_t \ \mathcal{F}_{(X^1,\dots,X^k)}[t]) B, \qquad (160)
$$

where $\mathcal{F}_{(X^1,\dots,X^k)}$ and $\mathcal{G}_{(X^1,\dots,X^k)}$ are the multiform functionals induced by F and G, and $\alpha \in \mathbb{R}$ and $B \in \bigwedge V$.

3.7.2 The Operators $\partial_t *$

Let $(\{\varepsilon^j\}, \{\varepsilon_j\})$ be a pair of arbitrary reciprocal basis for V and let $*$ be any one of the multiform products as before. We define

$$
\partial_t * \mathcal{F}_{(X^1,\dots,X^k)}[t] := \frac{1}{p!} \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_p} * \mathcal{F}'_{(X^1,\dots,X^k)\varepsilon_{j_1} \wedge \dots \wedge \varepsilon_{j_p}}[t]. \qquad (161)
$$

[34] Note that in [65] we used the notation $\partial_{t(A)}$ in the place of $A \cdot \partial_t$ and called that operator the directional derivative of the functional $\mathcal{F}_{(X^1,\dots,X^k)}[t]$ in the direction of A. We think now that the new denomination is more appropriated, something that will become clear when we calculate some examples.

Observe first that $\partial_t * \mathcal{F}_{(A^1,\ldots,A^k)}[t]$ does not depend on the pair of reciprocal basis and that taking into account Eq.(155) and Eq.(153) we have

$$\begin{aligned}
\partial_t * \mathcal{F}_{(X^1,\ldots,X^k)}[t] &= \frac{1}{p!}\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_p} * \sum_{i=1}^{k}(\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_p}) \cdot X^i (\partial_{X^i} F)_{(X^1,\ldots,X^k)}[t] \\
&= \sum_{i=1}^{k}\frac{1}{p!}(\varepsilon_{j_1} \wedge \ldots \wedge \varepsilon_{j_p}) \cdot X^i (\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_p}) * (\partial_{X^i} F)_{(A^1,\ldots,A^k)}[t] \\
&= \sum_{i=1}^{k} X^i * (\partial_{X^i} F)_{(X^1,\ldots,X^k)}[t] \\
&= \sum_{i=1}^{k} X^i * \partial_{t(X^k)} F[t(X^1),\ldots,t(X^k)]. \qquad (162)
\end{aligned}$$

We call the objects $\partial_t \wedge \mathcal{F}_{(A^1,\ldots,A^k)}[t]$, $\partial_t \cdot \mathcal{F}_{(A^1,\ldots,A^k)}[t]$, $\partial_t \lrcorner \mathcal{F}_{(A^1,\ldots,A^k)}[t]$ and $\partial_t \mathcal{F}_{(A^1,\ldots,A^k)}[t]$ respectively the *rotational*, the divergence, the left contracted divergence *and the gradient* (or simply the derivative) of $\mathcal{F}_{(X^1,\ldots,X^k)}$ with respect to t [65].

3.7.3 Examples

We present now some illustrative examples of derivatives of multiform functionals that will be need in Section 5 (and Appendix C) where we shall derive the equations of motion of the gravitational field from the variational principle.

Example 3.7 Let $\boldsymbol{h} \in (1,1)\text{-}extV$ and consider the scalar functional $\boldsymbol{h} \mapsto \boldsymbol{h}(b) \cdot \boldsymbol{h}(c)$ and the biform functional $\boldsymbol{l} \mapsto \boldsymbol{l}(b) \wedge \boldsymbol{l}(c)$, with $b, c \in \bigwedge^1 V$. The first has as generator function the scalar function of two 1-form variables $(x, y) \mapsto x \cdot y$ and the generator of the second functional is the biform function of two 1-form variables $(x, y) \mapsto x \wedge y$. Let us calculate the derivative of those functionals, with respect to $\boldsymbol{h}$ in the direction of $a \in \bigwedge^1 V$.

$$\begin{aligned}
a \cdot \partial_{\boldsymbol{h}}(\boldsymbol{h}(b) \cdot \boldsymbol{h}(c)) &= a \cdot b\partial_{\boldsymbol{h}(b)}(\boldsymbol{h}(b) \cdot \boldsymbol{h}(c)) + a \cdot c\partial_{\boldsymbol{h}(c)}(\boldsymbol{h}(b) \cdot \boldsymbol{h}(c)) \\
&= a \cdot b\boldsymbol{h}(c) + a \cdot c\boldsymbol{h}(b), \qquad (163)
\end{aligned}$$

were we utilized $\partial_x(x \cdot y) = y$.

Also

$$\begin{aligned}
a \cdot \partial_{\boldsymbol{h}}(\boldsymbol{h}(b) \wedge \boldsymbol{h}(c)) &= a \cdot b\partial_{\boldsymbol{h}(b)}(\boldsymbol{h}(b) \wedge \boldsymbol{h}(c)) + a \cdot c\partial_{\boldsymbol{h}(c)}(\boldsymbol{h}(b) \wedge \boldsymbol{h}(c)) \\
&= a \cdot b(n-1)\boldsymbol{h}(c) - a \cdot c(n-1)\boldsymbol{h}(b) \\
&= (n-1)(a \cdot b\boldsymbol{h}(c) - a \cdot c\boldsymbol{h}(b)) \\
&= (n-1)\boldsymbol{h}(a \cdot bc - a \cdot cb) \\
&= (n-1)\boldsymbol{h}(a \lrcorner (b \wedge c)), \qquad (164)
\end{aligned}$$

where we used $\partial_x(x \wedge y) = (n-1)y$.

Eq.(164) has a very useful generalization. The derivative of the k-form functional $\boldsymbol{h} \mapsto \underline{\boldsymbol{h}}(a^1 \wedge \ldots \wedge a^k) = \boldsymbol{h}(a^1) \wedge \ldots \wedge \boldsymbol{h}(a^k)$, with $a^1, \ldots, a^k \in \bigwedge^1 V$ is

$$a \cdot \partial_{h} \underline{\boldsymbol{h}}(a^1 \wedge \ldots \wedge a^k) = (n - k + 1)\underline{\boldsymbol{h}}(a \lrcorner (a^1 \wedge \ldots \wedge a^k)). \tag{165}$$

Example 3.8 Let $t \in (1,1)$-$extV$. The trace of t (i.e., $t \mapsto \mathrm{tr}[t] = t(\varepsilon^j) \cdot \varepsilon_j$) is a scalar functional of t whose generator is the scalar function of n 1-form variables, $(x^1, \ldots, x^n) \mapsto x^1 \cdot \varepsilon_1 + \cdots + x^n \cdot \varepsilon_n$, and $\mathrm{bif}[t] = t(\varepsilon^j) \wedge \varepsilon_j)$ is the biform functional of t whose generator is the biform function of n 1-form variables, $(x^1, \ldots, x^n) \mapsto x^1 \wedge \varepsilon_1 + \cdots + x^n \wedge \varepsilon_n$. Let us calculate $a \cdot \partial_t \mathrm{tr}[t]$ and $\partial_{\boldsymbol{h}(a)} \mathrm{bif}[t]$. We have

$$\begin{aligned} a \cdot \partial_{h} \mathrm{tr}[t] &= a \cdot \varepsilon^1 \partial_{t(\varepsilon^1)}(t(\varepsilon^1) \cdot \varepsilon_1 + \cdots + t(\varepsilon^n) \cdot \varepsilon_n) + \cdots \\ &\quad + a \cdot \varepsilon^n \partial_{t(\varepsilon^n)}(t(\varepsilon^1) \cdot \varepsilon_1 + \cdots + t(\varepsilon^n) \cdot \varepsilon_n) \\ &= a \cdot \varepsilon^1 \varepsilon_1 + \cdots + a \cdot \varepsilon^n \varepsilon_n = a, \end{aligned} \tag{166}$$

and

$$\begin{aligned} a \cdot \partial_{h} \mathrm{bif}[t] &= a \cdot \varepsilon^1 \partial_{t(\varepsilon^1)}(t(\varepsilon^1) \wedge \varepsilon_1 + \cdots + t(\varepsilon^n) \wedge \varepsilon_n) + \cdots \\ &\quad + a \cdot \varepsilon^n \partial_{t(\varepsilon^n)}(t(\varepsilon^1) \wedge \varepsilon_1 + \cdots + t(\varepsilon^n) \wedge \varepsilon_n) \\ &= a \cdot \varepsilon^1 (n-1)\varepsilon_1 + \cdots + a \cdot \varepsilon^n (n-1)\varepsilon_n = (n-1)a. \end{aligned} \tag{167}$$

Example 3.9 Let $\boldsymbol{h} \in (1,1)$-$extV$, and consider the 1-form functional $\boldsymbol{h} \mapsto \boldsymbol{h}^\dagger(b)$, with $b \in \bigwedge^1 V$, and the pseudo-scalar functional $\boldsymbol{h} \mapsto \underline{\boldsymbol{h}}(\tau)$, with $\tau \in \bigwedge^n V$. Let us calculate the derivatives of those functionals.

$$\begin{aligned} a \cdot \partial_{h} \boldsymbol{h}^\dagger(b) &= a \cdot \partial_{h}(\boldsymbol{h}^\dagger(b) \cdot \varepsilon^k \varepsilon_k) = a \cdot \partial_{h}(b \cdot \boldsymbol{h}(\varepsilon^k)\varepsilon_k) \\ &= a \cdot \varepsilon^1 \partial_{h(\varepsilon^1)}(b \cdot \boldsymbol{h}(\varepsilon^k)\varepsilon_k) + \cdots + a \cdot \varepsilon^n \partial_{h(\varepsilon^n)}(b \cdot \boldsymbol{h}(\varepsilon^k)\varepsilon_k) \\ &= a \cdot \varepsilon^1 (b\varepsilon_1) + \cdots + a \cdot \varepsilon^n (b\varepsilon_n) \\ &= b(a \cdot \varepsilon^1 \varepsilon_1 + \cdots + a \cdot \varepsilon^n \varepsilon_n) = ba, \end{aligned} \tag{168}$$

where we utilized a formula for the expansion of 1-forms in order to get $\boldsymbol{h}^\dagger(b)$ as a 1-form functional of $\boldsymbol{h}$, the scalar product condition involving $\boldsymbol{h}$ and $\boldsymbol{h}^\dagger$ (i.e., $\boldsymbol{h}^\dagger(x) \cdot y = x \cdot \boldsymbol{h}(y)$) and the formula $\partial_x (b \cdot x)c = bc$.

Also,

$$\begin{aligned} a \cdot \partial_{h} \underline{\boldsymbol{h}}(\tau) &= a \cdot \partial_{h} \underline{\boldsymbol{h}}((\tau \cdot \varepsilon_1 \wedge \ldots \wedge \varepsilon_n)\varepsilon^1 \wedge \ldots \wedge \varepsilon^n) \\ &= (\tau \cdot \varepsilon_1 \wedge \ldots \wedge \varepsilon_n) a \cdot \partial_{h} \underline{\boldsymbol{h}}(\varepsilon^1 \wedge \ldots \wedge \varepsilon^n) \\ &= (\tau \cdot \varepsilon_1 \wedge \ldots \wedge \varepsilon_n) \underline{\boldsymbol{h}}(a \lrcorner (\varepsilon^1 \wedge \ldots \wedge \varepsilon^n)) \\ &= \underline{\boldsymbol{h}}(a \lrcorner (\tau \cdot \varepsilon_1 \wedge \ldots \wedge \varepsilon_n)\varepsilon^1 \wedge \ldots \wedge \varepsilon^n) \\ &= \underline{\boldsymbol{h}}(a \lrcorner \tau) = \underline{\boldsymbol{h}}(a\tau), \end{aligned} \tag{169}$$

where we utilized the formula for expansion of pseudo-scalars, Eq.(159) and Eq.(165).

Example 3.10 Let $\boldsymbol{h} \in (1,1)\text{-}extV$, the determinant of $\boldsymbol{h}$ is a well defined scalar functional, $\boldsymbol{h} \mapsto \det[\boldsymbol{h}] = \underline{h}(\varepsilon^1 \wedge \ldots \wedge \varepsilon^n) \cdot (\varepsilon_1 \wedge \ldots \wedge \varepsilon_n)$. Let us calculate $\partial_{h(a)} \det[\boldsymbol{h}]$.

$$\begin{aligned} a \cdot \partial_h \det[\boldsymbol{h}] &= a \cdot \partial_h(\underline{\boldsymbol{h}}(\tau)\tau) = (a \cdot \partial_{\boldsymbol{h}} \underline{\boldsymbol{h}}(\tau))\tau^{-1} \\ &= \underline{\boldsymbol{h}}(a\tau)\tau^{-1} = \det[\boldsymbol{h}]\boldsymbol{h}^{\clubsuit}(a), \end{aligned} \tag{170}$$

where we utilized Eq.(160), Eq.(169) and the following identities involving (1,1)-extensors: $\det[t]\tau = \underline{t}(\tau)$ and $t^{-1}(a) = \det^{-1}[t]\underline{t}^{\dagger}(a\tau)\tau^{-1}$, (recall that $\boldsymbol{h}^{\clubsuit} = (\boldsymbol{h}^{\dagger})^{-1} = (\boldsymbol{h}^{-1})^{\dagger}$).

3.8 The Variational Operator $\boldsymbol{\delta}_t^w$

Let $\mathcal{F}_{(X^1,\ldots,X^k)}[t]$ be a scalar functional of the extensor variable $t : \bigwedge^1 V \to \bigwedge^1 V$. Let also $w : \bigwedge^1 V \to \bigwedge^1 V$. Construct next the real variable function $f_w(\lambda) = \mathcal{F}_{(X^1,\ldots,X^k)}[t + \lambda w]$. Then according to the mean value theorem we have

$$f_w(\lambda) = f_w(0) + \frac{d}{d\lambda} f_w(\lambda)|_{\lambda=0}\, \lambda + \frac{1}{2!} \frac{d^2}{d\lambda^2} f_w(\lambda)|_{\lambda=\lambda_1} \lambda^2, \qquad 0 < |\lambda_1| < |\lambda|. \tag{171}$$

The functional variation of $\mathcal{F}_{(X^1,\ldots,X^k)}[t]$ is by definition:

$$\boldsymbol{\delta}_t^w \mathcal{F}_{(X^1,\ldots,X^k)}[t] := \frac{d}{d\lambda} \mathcal{F}_w(\lambda)|_{\lambda=0}. \tag{172}$$

Since the Lagrangian for the field theory of gravitation (to be developed in Section 5) is a scalar functional of a (1,1)-extensor field $\boldsymbol{h}$, its functional variation will play an important role in this paper. In particular we shall need the following result.

Example 3.11 Calculate $\boldsymbol{\delta}_t^w \det[t]$. We first construct the real variable function

$$f_w(\lambda) = \det[t + \lambda w] \tag{173}$$

Next we define $g = tt^{+}$ and take a pair of reciprocal basis $(\{\varepsilon^i\}, \{\varepsilon_i\})$ for V satisfying the condition

$$g(\varepsilon^i) \cdot \varepsilon^j = \delta^{ij}, \tag{174}$$

which implies that

$$t(\varepsilon^i) \cdot t(\varepsilon^j) = \delta^{ij}. \tag{175}$$

Recalling Eq.(59a) we can write

$$f_w(\lambda) = \frac{1}{n!}\{(t(\varepsilon^1) + \lambda w(\varepsilon^1)] \wedge \ldots \wedge [t(\varepsilon^n) + \lambda w(\varepsilon^n)]\} \lrcorner (\varepsilon_1 \wedge \ldots \wedge \varepsilon_n). \tag{176}$$

Then we have

$$\frac{d}{d\lambda}\, f_w(\lambda)|_{\lambda=0} = \frac{1}{n!}\{[w(\varepsilon^1)\wedge t(\varepsilon^2)\wedge ...t(\varepsilon^n)]\lrcorner(\varepsilon_1\wedge ...\wedge\varepsilon_n)$$
$$+ ..... + [t(\varepsilon^1)\wedge t(\varepsilon^2)\wedge ...\wedge w(\varepsilon^n)]\lrcorner(\varepsilon_1\wedge ...\wedge\varepsilon_n)\} \tag{177}$$

and recalling the identity giving by Eq.(35), which says that for any $A, B, C \in \bigwedge V$, it is $(A\wedge B)\lrcorner C = A\lrcorner(B\lrcorner C)$, and Eq.(65) $(t^{-1}(a) = \det^{-1}[t]\underline{t}^{\dagger}(a\tau)\tau^{-1})$, we can write the second member of Eq.(177) as

$$\begin{aligned}
&\frac{1}{n!}\{w(\varepsilon^1)\lrcorner(t(\varepsilon^2)\wedge ...\wedge t(\varepsilon^n))\lrcorner(\varepsilon_1\wedge ...\varepsilon_n) + ...\\
&+ (-1)^{n-1}w(\varepsilon^n)\lrcorner(t(\varepsilon^1)\wedge ...\wedge t(\varepsilon^{n-1}))\lrcorner(\varepsilon_1\wedge ...\wedge\varepsilon_n)\}\\
&= \frac{1}{n!}\{w(\varepsilon^1)\lrcorner[t(\varepsilon^1)\lrcorner(t(\varepsilon^1)(t(\varepsilon^2)\wedge ...\wedge t(\varepsilon^n))\lrcorner(\varepsilon_1\wedge ...\wedge\varepsilon_n)...\\
&+ w(\varepsilon^n)\lrcorner[t(\varepsilon^n)\lrcorner(t(\varepsilon^1)(t(\varepsilon^2)\wedge ...\wedge t(\varepsilon^n))\lrcorner(\varepsilon_1\wedge ...\wedge\varepsilon_n)\}\\
&= \sum_{i=1}^{n} w(\varepsilon^i)\lrcorner[t(\varepsilon^i)\lrcorner(t(\varepsilon^1)(t(\varepsilon^2)\wedge ...\wedge t(\varepsilon^n))\lrcorner(\varepsilon_1\wedge ...\wedge\varepsilon_n)\\
&= \sum_{i=1}^{n} w(\varepsilon^i)\lrcorner t(\varepsilon^i\tau)\tau^{-1}\\
&= \sum_{i=1}^{n} w(\varepsilon^i)\lrcorner t^{\clubsuit}(\varepsilon^i)\det[t]\\
&= w(\partial_a)\lrcorner t^{\clubsuit}(a)\det[t].
\end{aligned} \tag{178}$$

Taking moreover into account that for any $a \in \bigwedge^1 V$,

$$\sum_{j=1}^{n} w(\varepsilon^i)\lrcorner t^{\clubsuit}(\varepsilon^i)\det[t] = w(\partial_a)\lrcorner t^{\clubsuit}(a)\det[t], \tag{179}$$

we finally get

$$\boldsymbol{\delta}_t^w \det[t] = w(\partial_a)\lrcorner t^{\clubsuit}(a)\det[t]. \tag{180}$$

4 Multiform and Extensor Calculus on Manifolds

4.1 Canonical Space

Let M be a smooth (i.e., C^∞) differential manifold, $\dim M = n$. Take an arbitrary point $O \in M$, and a local chart $(\mathcal{U}, \phi)_o$ of an atlas of M such that $O \in \mathcal{U}$.

Any point $p \in \mathcal{U}$ is then localized by a n-uple of real numbers $\phi(p) \in \mathbb{R}^n$, say $\phi(p) = (\xi^1(p), \ldots, \xi^n(p))$. As usual both $\mathcal{U} \ni p \mapsto \boldsymbol{\xi}^\mu(p) \in \mathbb{R}$, the μ-th

coordinate function as well $\boldsymbol{\xi}^\mu(p) = \xi^\mu$, the μ-th coordinate , are denoted (when no confusion arises) by the same notation ξ^μ. Moreover, here $\mu = 1, \ldots, n$.

At $p \in \mathcal{U}$, the set of coordinate tangent vectors $\left\{ \left.\frac{\partial}{\partial \xi^1}\right|_{(p)}, \ldots, \left.\frac{\partial}{\partial \xi^n}\right|_{(p)} \right\}$ is a natural basis for the tangent space T_pM, and the set of tangent coordinate 1-forms $\left\{ \left.d\xi^1\right|_{(p)}, \ldots, \left.d\xi^n\right|_{(p)} \right\}$ is a natural basis for the cotangent (or dual) space T_p^*M.

We introduce an equivalence relation on the (sub)tangent bundle $T\mathcal{U} = \bigcup_{p\in\mathcal{U}} T_pM$ as follows. Let $\mathbf{v}_p \in T_pM$ and $\mathbf{v}_q \in T_qM$. We say that $\mathbf{v}_p \sim \mathbf{v}_q$ iff $\mathbf{v}_p\xi^\mu = \mathbf{v}_q\xi^\mu$, with $\mu = 1, \ldots, n$.

This equivalence relation is well defined and, of course, it is not void, since the coordinate tangent vectors at any two points, $p, q \in \mathcal{U}$ are equivalent. Indeed, we have

$$\left.\frac{\partial}{\partial \xi^\alpha}\right|_{(p)} \xi^\mu = \left.\frac{\partial \xi^\mu}{\partial \xi^\alpha}\right|_{(p)} = \delta^\mu_\alpha = \left.\frac{\partial \xi^\mu}{\partial \xi^\alpha}\right|_{(q)} = \left.\frac{\partial}{\partial \xi^\alpha}\right|_{(q)} \xi^\mu, \quad \mu, \alpha = 1, \ldots, n, \tag{181}$$

and thus $\left.\frac{\partial}{\partial \xi^\alpha}\right|_{(p)} \sim \left.\frac{\partial}{\partial \xi^\alpha}\right|_{(q)}$.

For the point $O \in \mathcal{U}$, the equivalence class of a tangent vector $\mathbf{v}_o \in T_oM$ is given by $[\mathbf{v}_o] = \{\mathbf{v}_p \sim \mathbf{v}_o \mid p \in \mathcal{U}\}$.

The set of the equivalence classes of each one of the tangent vectors $\mathbf{v}_o \in T_oM$, i.e., $\mathbf{U} = \{[\mathbf{v}_o] \,/\, \mathbf{v}_o \in T_oM\}$, equipped with the *sum* of equivalence classes and the *multiplication* of equivalence classes by scalars (real numbers), defined by

$$[\mathbf{v}_o] + [\mathbf{w}_o] = [\mathbf{v}_o + \mathbf{w}_o]; \qquad \alpha[\mathbf{v}_o] = [\alpha\mathbf{v}_o], \tag{182}$$

has a natural structure of vector space over $\mathbb{R}$.

The equivalence class of the null vector $\mathbf{0}_o \in \mathcal{T}_oM$, i.e., $\mathbf{0} \equiv [\mathbf{0}_o] \in \mathbf{U}$, i.e., the set of all null vectors in $T\mathcal{U}$ is the null vector of $\mathbf{U}$.

The set of the equivalence classes of the n-tangent coordinate vectors $\left\{ \left.\frac{\partial}{\partial \xi^i}\right|_{(o)} \right\}$, with $\left.\frac{\partial}{\partial \xi^i}\right|_{(o)} \in T_oM$, say $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ with

$$\mathbf{b}_1 \equiv [\left.\frac{\partial}{\partial \xi^1}\right|_{(o)}], \ldots, \mathbf{b}_n \equiv [\left.\frac{\partial}{\partial \xi^n}\right|_{(o)}] \in \mathbf{U}, \tag{183}$$

is a basis for $\mathbf{U}$ called the *fiducial basis* and, of course, $\dim \mathbf{U} = n$.

$\mathbf{U}$ is a vector space obviously associated to the chart $(\mathcal{U}, \phi)_o$, but nevertheless it will be said to be the canonical space since it will play a fundamental role in our theory of multiform and extensor fields on manifolds. To continue we recall that from the above identifications we have immediately the identification of $\bigwedge T_p^*M$ with $\bigwedge T_q^*M$ for all $p, q \in \mathcal{U}$.

Taking into account the notations introduced in Section 2 the dual space of $\mathbf{U}$ is denoted U, the space of k-forms is denoted by $\bigwedge^k U$ and the space of multiforms is denoted by $\bigwedge U$.

The dual basis of the fiducial basis $\{\mathbf{b}_\mu\}$ is said to be a fiducial basis for U and is denoted $\{\boldsymbol{\beta}^\mu\}$, i.e., $\boldsymbol{\beta}^\mu(\mathbf{b}_\nu) = \delta^\mu_\nu$. To be able to use Einstein's sum convention we introduce also the notation,

$$b^\mu = \mathbf{b}_\mu \ e \ \beta_\mu = \boldsymbol{\beta}^\mu. \tag{184}$$

The canonical scalar product is constructed as in Section 2 using $\{\mathbf{b}_\mu\}$ and in what follows we use for our calculations the canonical Clifford algebra $\mathcal{C}\ell(U, \cdot)$.

4.1.1 The Position 1-Form

To any $p \in \mathcal{U}$ there correspond exactly n real numbers $\xi^1, \ldots, \xi^n$, its position coordinates in the local chart $(\mathcal{U}, \phi)_o$. It is thus possible to define a 1-form on U,

$$x = \xi^\mu \beta_\mu, \tag{185}$$

associated to $p \in \mathcal{U}$. The 1-form $x \in U$ *localize* p in the vector space U and we call x the *position 1-form* (*or position vector*) *of* p.

Observe that given an arbitrary 1-form on U, it will not be necessarily the position vector of some point of the open set $\mathcal{U}$, since the components of an arbitrary element of U does not need to be necessarily the coordinates of some point of $\mathcal{U}$.

The set of all position vectors of the points of $\mathcal{U}$ is denoted $U_o \subset U$. This subset of the vector space U is not necessarily a vector subspace of U.

Observe also that we have immediately for any $x, y \in U_o$ the identification of the tangent spaces $\bigwedge T_x^* U$ and $\bigwedge T_y^* U$ with $\bigwedge U$.

4.2 Multiform Fields

From what has been said it is obvious that any $\mathcal{X} \in \sec \bigwedge T^* M$ when restricted to $\mathcal{U}$ will be represented by a multiform function X of the position vector, which we will write as

$$X \in \sec \bigwedge T^* U \tag{186}$$

meaning that for each $x \in U_o$, $X(x) = X_{(x)} \in \bigwedge T_x^* U \simeq \bigwedge U$. Eventually we also use the notation $X : U_0 \to \bigwedge U$.

If $X \in \sec \bigwedge T^* U$ is C^∞-differentiable on the set U_0, then the multiform $\mathcal{X} \in \sec \bigwedge T^* M$ is differentiable on $\mathcal{U}$.

4.2.1 Extensor Fields

Any (p,q)-extensor field $\mathfrak{t}$ on M, when restricted to $\mathcal{U}$ is represented by an extensor function t of the position vector. We write

$$t \in \sec(p,q)\text{-}extU, \tag{187}$$

and eventually also write $t : U_0 \to (p,q)\text{-}extU$.

If t is C^∞-differentiable[35] on U_o, then the (p,q)-extensor field $\mathfrak{t}$ is smooth on $\mathcal{U}$. For a general extensor field, say Δ we use the notation $\Delta \in \sec extU$.

4.3 Parallelism Structure (U_0, λ) and Covariant Derivatives

Given a smooth manifold M ($\dim M = n$) and an *arbitrary* linear connection ∇ on M, the pair (M, ∇) is said to be a *parallelism structure*. We will now analyze some important parallelism structures that will appear in the next sections.

4.3.1 The Connection 2-*extensor field* γ on U_o and Associated Extensors Covariant Derivative of Multiform Fields Associated to (U_0, γ)

The parallelism structure (M, ∇) is represented on U_o by a pair (U_0, γ) (also called a parallelism structure) where $\gamma \in \sec extU$ is called a smooth *connection* 2-*extensor field* on U_o, i.e., for each $x \in U_0$,

$$\gamma_{(x)} : \bigwedge^1 T_xU \times \bigwedge^1 T_xU \to \bigwedge^1 T_xU. \tag{188}$$

We also define the following fields associated to γ. First, given $a \in \sec \bigwedge^1 T^*U$, define the smooth $(1,1)$-extensor field on U_o, $\gamma_a \in \sec(1,1)\text{-}extU$ by

$$\gamma_a(b) = \gamma(a,b) \tag{189}$$

for all $b \in \sec \bigwedge^1 T^*U$. The field γ_a is called a *connection* $(1,1)$-*extensor field on* U_o.

Next we introduce a smooth $(1,2)$-extensor field on U_0, say $\omega \in \sec(1,2)$-$extU$ such that for $a \in \sec \bigwedge^1 T^*U$, $\omega(a) \in \sec \bigwedge^2 T^*U$ is given by

$$\omega(a) := \frac{1}{2}\mathrm{bif}[\gamma_a]. \tag{190}$$

The field ω will be called (for reasons that will become clear in a while) a *rotation gauge field.*

Finally we define the smooth extensor field on U_o, $\Gamma_a \in \sec extU$ such that for each $x \in U_o$, $\Gamma_{(x)a}$ is the generalized of $\gamma_{(x)a}$ (recall Eq.(70)).

[35] A (p,q)-extensorial function t is said to be C^∞-differentiable on U_0 iff for an any C^∞-differentiable p-form function on U_0, say $x \mapsto X(x)$, the q-form function $x \mapsto t_{(x)}(X(x))$ is C^∞-differentiable on U_0.

4.3.2 Covariant Derivative of Multiform Fields Associated to (U_0, γ)

The connection 2-extensor field γ is used in the following way. For any smooth $a \in \sec \bigwedge^1 T^*U$ which is a representative on U of a vector field $\mathfrak{a} \in \sec T\mathcal{U}$ we introduce two covariant operators ∇_a and ∇_a^-, acting the module of smooth multiform fields on U_o.

First, if $b \in \sec \bigwedge^1 T^*U$

$$\nabla_a b := a \cdot \partial b + \gamma_a(b), \tag{191a}$$

$$\nabla_a^- b := a \cdot \partial b - \gamma_a(b) \tag{191b}$$

For any smooth $X \in \sec \bigwedge T^*U$ we define

$$\nabla_a X := a \cdot \partial X + \Gamma_a(X), \tag{192a}$$

$$\nabla_a^- X := a \cdot \partial X - \Gamma_a^\dagger(X), \tag{192b}$$

where Γ_a is the generalized of γ_a and $\Gamma_a^\dagger$ is the adjoint of Γ_a.

The operators ∇_a and ∇_a^- are well defined covariant derivatives on U_o and thus satisfy all properties of a covariant derivative operator, i.e., they are linear with respect to the direction 1-form field, i.e., we have

$$\nabla^{\pm}_{\alpha a+\beta b} X = \alpha \nabla^{\pm}_a X + \beta \nabla^{\pm}_b X, \tag{193}$$

for $\alpha, \beta \in \mathbb{R}$ and $a, b \in \sec \bigwedge^1 T^*U$ and

$$\nabla_a f = a \cdot \partial f, \;\; \nabla_a(X+Y) = \nabla_a X + \nabla_a Y, \;\; \nabla_a(fX) = a \cdot \partial f + f \nabla_a X, \tag{194}$$

with analog equations for ∇_a^-.

To prove, e.g., that $\nabla_a f = a \cdot \partial f$ it is enough to recall a property of the generalized T of a given $(1,1)$-extensor t, explicitly $T(\alpha) = 0$, with $\alpha \in \mathbb{R}$. We then have using Eq.(192a) that for any smooth scalar field f indeed $\nabla_a f = a \cdot \partial f + \Gamma_a(f) = a \cdot \partial f$.

The proofs of the remaining formulas in Eq.(194) are equally elementary.

We can prove also that ∇_a and ∇_a^- satisfy *Leibniz rule* for the exterior product of smooth multiform fields, i.e.,

$$\nabla_a(X \wedge Y) = (\nabla_a X) \wedge Y + X \wedge (\nabla_a Y), \tag{195}$$

with an analogous formula for ∇_a^-. Indeed, take two smooth multiform fields X and Y. Utilizing the Leibniz rule satisfied by the directional derivative operator $a \cdot \partial$ when applied to the exterior product of smooth multiform fields and a property of the generalized of a $(1,1)$-extensor t (namely $T(A \wedge B) = T(A) \wedge B + A \wedge T(B)$, for any $A, B \in \sec \bigwedge T^*U$), we get

$$\begin{aligned}\nabla_a(X \wedge Y) &= a \cdot \partial(X \wedge Y) + \Gamma_a(X \wedge Y)\\ &= (a \cdot \partial X) \wedge Y + X \wedge (a \cdot \partial Y) + \Gamma_a(X) \wedge Y + X \wedge \Gamma_a(Y)\\ \nabla_a(X \wedge Y) &= (\nabla_a X) \wedge Y + X \wedge (\nabla_a Y),\end{aligned} \tag{196}$$

which proves Eq.(195).

On the other hand the operator of ordinary directional derivative $a\cdot\partial$ and the operators ∇_a and ∇_a^- are related by a notable identity:

$$a\cdot\partial(X\cdot Y)=(\nabla_a X)\cdot Y+X\cdot(\nabla_a^- Y)=(\nabla_a^- X)\cdot Y+X\cdot(\nabla_a Y). \tag{197}$$

Indeed, take two smooth multiform fields X and Y. Utilizing the Leibniz rule for $a\cdot\partial$ when applied to the exterior product of smooth multiform fields and that $T(A)\cdot B=A\cdot T^{\dagger}(B)$, for all $A,B\in\sec\bigwedge T^*U$we see immediately the validity of the formulas

$$\begin{aligned}(\nabla_a X)\cdot Y+X\cdot(\nabla_a^- Y)&=(a\cdot\partial X)\cdot Y+\Gamma_a(X)\cdot Y+X\cdot(a\cdot\partial Y)-X\cdot\Gamma_a^{\dagger}(Y)\\&=(a\cdot\partial X)\cdot Y+X\cdot(a\cdot\partial Y)\\(\nabla_a X)\cdot Y+X\cdot(\nabla_a^- Y)&=a\cdot\partial(X\cdot Y).\end{aligned} \tag{198}$$

4.3.3 Covariant Derivative of Extensor Fields Associated to (U_0,γ)

The covariant derivative operators ∇_a and ∇_a^- may be extended to act on the *module* of smooth (p,q)-extensor fields on U_o (which represent smooth (p,q)-extensor field on $\mathcal{U}\subset M$). We define for $t\in\sec(p,q)$-$extU$, $\nabla_a t$ and $\underset{p}{\nabla_a^-} t$ as the smooth (p,q)-extensor fields on U_o such that for any smooth $X\in\sec\bigwedge T^*U$,

$$(\nabla_a t)(X):=\nabla_a t(X)-t(\nabla_a^- X), \tag{199a}$$

$$(\nabla_a^- t)(X):=\nabla_a^- t(X)-t(\nabla_a X). \tag{199b}$$

Observe that in the above formulas $\nabla_a t(X)$ denotes the covariant derivative ∇_a of the smooth multiform field $t(X)$ and $\nabla_a^- X$ is the covariant derivative ∇_a^- of the smooth multiform field X.

Certainly those properties are consistent with the linearity property of the smooth (p,q)-extensor fields. Indeed, take two smooth p-form fields X and Y, and a smooth scalar field f. We have immediately

$$\begin{aligned}(\nabla_a t)(X+Y)&=\nabla_a t(X+Y)-t(\nabla_a^-(X+Y))\\&=\nabla_a(t(X)+t(Y))-t(\nabla_a^- X+\nabla_a^- Y)\\&=\nabla_a t(X)+\nabla_a t(Y)-t(\nabla_a^- X)-t(\nabla_a^- Y)\\&=(\nabla_a t)(X)+(\nabla_a t)(Y),\end{aligned} \tag{200}$$

and

$$\begin{aligned}(\nabla_a t)(fX)&=\nabla_a t(fX)-t(\nabla_a^-(fX))=\nabla_a(ft(X))-t((a\cdot\partial f)X+f\nabla_a^- X)\\&=(a\cdot\partial f)t(X)+f\nabla_a t(X)-(a\cdot\partial f)t(X)-ft(\nabla_a^- X)=f(\nabla_a t)(X).\end{aligned} \tag{201}$$

Each one of the covariant derivatives $t\mapsto\nabla_a t$ and $t\mapsto\nabla_a^- t$ possess the property of linearity with relation to the direction 1-form, i.e., $\nabla_{\alpha a+\beta b}^{\pm}t=\alpha\nabla_a^{\pm}t+\beta\nabla_b^{\pm}t$, where $\alpha,\beta\in\mathbb{R}$ and $a,b\in\sec\bigwedge^1 T^*U$

Moreover, $t \mapsto \nabla_a t$,and $t \mapsto \nabla_a^- t$ possess also the following properties:

$$\nabla_a^{\pm}(t+u) = \nabla_a^{\pm} t + \nabla_a^{\pm} u, \tag{202}$$
$$\nabla_a^{\pm}(ft) = (a \cdot \partial f)t + f\nabla_a^{\pm} t, \tag{203}$$

where t and u are smooth (p,q)-extensor fields and f is a scalar field.

Let us prove Eq.(202).and Eq.(203). Let X be an arbitrary smooth p-form field,then

$$\begin{aligned}(\nabla_a(t+u))(X) &= \nabla_a(t+u)(X) - (t+u)(\nabla_a^- X)\\ &= \nabla_a(t(X)+u(X)) - t(\nabla_a^- X) - u(\nabla_a^- X)\\ &= \nabla_a t(X) + \nabla_a u(X) - t(\nabla_a^- X) - u(\nabla_a^- X)\\ &= (\nabla_a t)(X) + (\nabla_a u)(X) = (\nabla_a t + \nabla_a u)(X),\end{aligned} \tag{204}$$

i.e., $\nabla_a(t+u) = \nabla_a t + \nabla_a u$. Also,

$$\begin{aligned}(\nabla_a(ft))(X) &= \nabla_a(ft)(X) - ft(\nabla_a X) = \nabla_a ft(X) - ft(\nabla_a X)\\ &= (a \cdot \partial f)t(X) + f\nabla_a t(X) - ft(\nabla_a X)\\ &= (a \cdot \partial f)t(X) + f(\nabla_a t)(X),\end{aligned} \tag{205}$$

i.e.,, $\nabla_a(ft) = (a \cdot \partial f)t + f(\nabla_a t)$.

4.3.4 Notable Identities

We end this section presenting two notable identities, which are:

(i) Let X be a smooth p-form field and Y a smooth q-form field, then

$$(\nabla_a t)(X) \cdot Y = a \cdot \partial(t(X) \cdot Y) - t(\nabla_a^- X) \cdot Y - t(X) \cdot \nabla_a^- Y. \tag{206}$$

Indeed, we have

$$\begin{aligned}(\nabla_a t)(X) \cdot Y &= \nabla_a t(X) \cdot Y - t(\nabla_a^- X) \cdot Y\\ &= a \cdot \partial t(X) \cdot Y + \Gamma_a(t(X)) \cdot Y - t(\nabla_a^- X) \cdot Y\\ &= a \cdot \partial t(X) \cdot Y + t(X) \cdot \Gamma_a^{\dagger}(Y) - t(\nabla_a^- X) \cdot Y\\ &= a \cdot \partial t(X) \cdot Y + t(X) \cdot a \cdot \partial Y - t(X) \cdot a \cdot \partial Y\\ &+ t(X) \cdot \Gamma_a^{\dagger}(Y) - t(\nabla_a^- X) \cdot Y\\ &= a \cdot \partial t(X) \cdot Y + t(X) \cdot a \cdot \partial Y - t(\nabla_a^- X) \cdot Y\\ &- t(X) \cdot (a \cdot \partial Y - \Gamma_a^{\dagger}(Y))\\ &= a \cdot \partial(t(X) \cdot Y) - t(\nabla_a^- X) \cdot Y - t(X) \cdot \nabla_a^- Y.\end{aligned}$$

(ii) For any smooth (p,q)-extensor field t, it is

$$\nabla_a t^{\dagger} = (\nabla_a t)^{\dagger}. \tag{207}$$

Take the smooth fields $X \in \sec \bigwedge^p T^*U$, and $Y \in \sec \bigwedge^q T^*U$. Utilizing twice Eq.(206) and the algebraic property $t(X) \cdot Y = X \cdot t^\dagger(Y)$, we have

$$\begin{aligned}(\nabla_a t^\dagger)(Y) \cdot X &= a \cdot \partial(t^\dagger(Y) \cdot X) - t^\dagger(\nabla_a^- Y) \cdot X - t^\dagger(Y) \cdot \nabla_a^- X \\ &= a \cdot \partial(t(X) \cdot Y) - t(X) \cdot \nabla_a^- Y - t(\nabla_a^- X) \cdot Y \\ &= (\nabla_a t)(X) \cdot Y = X \cdot (\nabla_a t)^\dagger(Y), \end{aligned} \tag{208}$$

which implies $(\nabla_a t^\dagger)(Y) = (\nabla_a t)^\dagger(Y)$, i.e., $\nabla_a t^\dagger = (\nabla_a t)^\dagger$.

For ∇_a^- completely analog properties hold, i.e., for any smooth p-form field X and all smooth q-form field Y it is

$$(\nabla_a^- t)(X) \cdot Y = a \cdot \partial(t(X) \cdot Y) - t(\nabla_a X) \cdot Y - t(X) \cdot \nabla_a Y. \tag{209}$$

Also, for any smooth (p, q)-extensor field t,

$$\nabla_a^- t^\dagger = (\nabla_a^- t)^\dagger. \tag{210}$$

4.3.5 The 2-Exform Torsion Field of the Structure (U_o, γ)

Given a parallelism structure (M, ∇) we know that the torsion and curvature operators (used for the definition of the torsion and Riemann curvature tensors) characterize ∇ completely. Let (U_o, γ) be the representative of the structure (M, ∇) restricted to $\mathcal{U}$. We now introduce the *representatives* of the torsion and curvature operators of (M, ∇) on U_o.

First we introduce the smooth torsion 2-*exform*[36] field $\overset{\gamma}{\tau} \in \sec \mathit{extU}$, such for smooth $a, b \in \sec \bigwedge^1 T^*U$ we have $(a, b) \mapsto \overset{\gamma}{\tau}(a, b) \in \sec \bigwedge^1 T^*U$, given by

$$\overset{\gamma}{\tau}(a, b) := \nabla_a b - \nabla_b a - [a, b] = \gamma_a(b) - \gamma_b(a). \tag{211}$$

Next we introduce the $(2, 1)$-extensor field $\overset{\gamma}{T} \in \sec(2, 1)$-$\mathit{extU}$,such that for any smooth $B \in \sec \bigwedge^1 T^*U$, $B \mapsto \overset{\gamma}{T}(B) \in \sec \bigwedge^1 T^*U$ we have:

$$\overset{\gamma}{T}(B) := \frac{1}{2} B \cdot (\partial_a \wedge \partial_b) \overset{\gamma}{\tau}(a, b). \tag{212}$$

$\overset{\gamma}{T}$ is called the *torsion* $(2, 1)$-*extensor field.*

Observe that if the covariant derivative ∇_a is *symmetric* (i.e., $\nabla_a b - \nabla_b a = [a, b]$, for all smooth 1-form fields a and b, then $\overset{\gamma}{\tau}(a, b) = 0$ and $\overset{\gamma}{T}(B) = 0$.

[36] A 2-exform on U is an antisymmetric 2-extensor on U , i.e., a linear mapping $\theta : \bigwedge^1 U \times \bigwedge^1 U \to \bigwedge U$ tsuch that for all $a, b \in \bigwedge^1 U$ it is $\theta(a, b) = -\theta(b, a)$.

4.4 Curvature Operator and Curvature Extensor Fields of the Structure (U_o, γ)

We introduce now a linear operator that acts on the Lie algebra of smooth 1-form fields on U_o. Given $a, b, c \in \sec \bigwedge^1 T^*U$ the curvature operator is the mapping

$$\begin{array}{cccc} \overset{\gamma}{\rho} & : & \sec(\bigwedge^1 T^*U \times \bigwedge^1 T^*U \times \bigwedge^1 T^*U) & \to & \sec \bigwedge^1 T^*U, \\ & & (a,b,c) & \mapsto & \overset{\gamma}{\rho}(a,b,c), \end{array}$$

such that

$$\overset{\gamma}{\rho}(a,b,c) := [\nabla_a, \nabla_b]c - \nabla_{[a,b]}c. \tag{213}$$

The operator $\overset{\gamma}{\rho}$ characterizes the curvature of the parallelism structure (M, ∇) on U_0. Its main properties are:

(i) $\overset{\gamma}{\rho}$ is antisymmetric with respect to the first and second variables, i.e.,

$$\overset{\gamma}{\rho}(a,b,c) = -\overset{\gamma}{\rho}(b,a,c). \tag{214}$$

(ii) If the covariant derivative ∇_a is symmetric, i.e., $\nabla_a b - \nabla_b a = [a,b]$ (or equivalently $\gamma_a(b) = \gamma_b(a)$, for all smooth 1-form fields a and b) then $\overset{\gamma}{\rho}$ has a cyclic property, i.e.,

$$\overset{\gamma}{\rho}(a,b,c) + \overset{\gamma}{\rho}(b,c,a) + \overset{\gamma}{\rho}(c,a,b) = 0. \tag{215}$$

The smooth scalar 4-extensor field[37], $(w,a,b,c) \mapsto \overset{\gamma}{\mathbf{R}}_1(w,a,b,c)$ such that

$$\overset{\gamma}{\mathbf{R}}_1(w,a,b,c) = w \cdot \overset{\gamma}{\rho}(b,c,a), \tag{216}$$

for all smooth 1-form fields w, a, b and c, is called the curvature 4-*extensor field*

The main properties of $\overset{\gamma}{\mathbf{R}}_1$ are:

(i) $\overset{\gamma}{\mathbf{R}}_1$ is antisymmetric with respect to the third and fourth variables, i.e.,

$$\overset{\gamma}{\mathbf{R}}_1(w,a,b,c) = -\overset{\gamma}{\mathbf{R}}_1(w,a,c,b). \tag{217}$$

(ii) If the covariant derivative ∇_a is symmetric then $\overset{\gamma}{\mathbf{R}}_1$ possess a cyclic property, i.e.,

$$\overset{\gamma}{\mathbf{R}}_1(w,a,b,c) + \overset{\gamma}{\mathbf{R}}_1(w,b,c,a) + \overset{\gamma}{\mathbf{R}}_1(w,c,a,b) = 0. \tag{218}$$

The smooth scalar 2-extensor field $(a,b) \mapsto \overset{\gamma}{\mathbf{R}}_2(a,b)$ such that

$$\overset{\gamma}{\mathbf{R}}_2(a,b) = \overset{\gamma}{\mathbf{R}}_1(\partial_w, a, w, b), \tag{219}$$

[37] A linear mapping $t : \sec(\bigwedge^1 U \times \bigwedge^1 U \times \bigwedge^1 U \times \bigwedge^1 U) \to \mathbb{R}$ is called a scalar 4-extensor.

for all smooth 1-form fields a and b, is called the *Ricci 2-extensor field*. Note that $\overset{\gamma}{\mathbf{R}}_2$ is an *internal contraction* of $\overset{\gamma}{\mathbf{R}}_1$ between the first and third variables calcualted as follows.. Let $(\{\varepsilon^\mu\},\{\varepsilon_\mu\})$ be a pair of reciprocal basis on U_0, then an internal contraction between the first and third variables is

$$\overset{\gamma}{\mathbf{R}}_1(\partial_w,a,w,b):=\frac{\partial}{\partial w_\mu}\overset{\gamma}{\mathbf{R}}_1(\varepsilon^\mu,a,w,b)=\overset{\gamma}{\mathbf{R}}_1(\varepsilon^\mu,a,\varepsilon_\mu,b)=\overset{\gamma}{\mathbf{R}}_1(\varepsilon_\mu,a,\varepsilon^\mu,b),$$

with μ summed from 0 to 3.

The smooth $(1,1)$-extensor field $b\mapsto\overset{\gamma}{\boldsymbol{R}}_1(b)$ such that

$$a\cdot\overset{\gamma}{\boldsymbol{R}}_1(b)=\overset{\gamma}{\mathbf{R}}_2(a,b), \tag{220}$$

for all smooth 1-form fields a and b, is called the *Ricci* $(1,1)$*-extensor field.*

Note that we can write $\overset{\gamma}{\boldsymbol{R}}_1(b)$ as

$$\overset{\gamma}{\boldsymbol{R}}_1(b)=\partial_a\overset{\gamma}{\mathbf{R}}_2(a,b), \tag{221}$$

once we recall that $\partial_a\overset{\gamma}{\mathbf{R}}_2(a,b)\equiv\varepsilon^\mu(\varepsilon_\mu\cdot\partial_a\overset{\gamma}{\mathbf{R}}_2(a,b))$, i.e., $\partial_a\overset{\gamma}{\mathbf{R}}_2(a,b)=\varepsilon^\mu\mathbf{R}_2(\varepsilon_\mu,b)$ due to the linearity of $\overset{\gamma}{\mathbf{R}}_2$ with respect to the first variable.

4.5 Covariant Derivatives Associated to Metric Structures $(U_o,\boldsymbol{g})$

4.5.1 Metric Structures

Let $\mathbf{g}$ be a smooth $(1,1)$-extensor field (associated to a metric tensor $\boldsymbol{g}\in\sec T_2^0M$). Its representative $\boldsymbol{g}$ on a given U_o is symmetric (i.e., $\boldsymbol{g}_{(x)}=\boldsymbol{g}_{(x)}^\dagger$ for all $x\in U_0$), is non degenerated (i.e., $\det[\boldsymbol{g}_{(x)}]\neq 0$ for $x\in U_0$). In what follows we say that $\boldsymbol{g}$ is a metric extensor field on U_0.

A pair $(M,\boldsymbol{g})$ is said to be a *metric structure* and its restriction to $\mathcal{U}\subset M$ is represented by $(U_o,\boldsymbol{g})$, and is also called a metric structure.

4.5.2 Christoffel Operators for the Metric Structure $(U_o,\boldsymbol{g})$

Let $\boldsymbol{g}$ be the representative of the extensor field $\mathbf{g}$ (associated to $\boldsymbol{g}$) on U_o.

The two classical Christoffel operators on $\mathcal{U}\subset M$ are represented on the structure $(U_o,\boldsymbol{g})$ by two operators that have the same name and that act on the Lie algebra of smooth 1-form fields on U_0. If $a,b,c\in\sec\bigwedge^1T^*U$ are such fields, we have

(i) *The first Christoffel operator* is the smooth mapping $(a,b,c)\mapsto[a,b,c]$ such that

$$\begin{aligned}[a,b,c]:=\frac{1}{2}(&a\cdot\partial(\boldsymbol{g}(b)\cdot c)+b\cdot\partial(\boldsymbol{g}(c)\cdot a)-c\cdot\partial(\boldsymbol{g}(a)\cdot b)\\&+\boldsymbol{g}(c)\cdot[a,b]+\boldsymbol{g}(b)\cdot[c,a]-\boldsymbol{g}(a)\cdot[b,c]),\end{aligned} \tag{222}$$

where $[a,b]$ is the Lie bracket of a and b defined by:

$$[a,b] := a \cdot \partial b - b \cdot \partial a. \tag{223}$$

(ii) *The second Christoffel operator*, $(a,b,c) \mapsto \left\{ \begin{matrix} c \\ a,b \end{matrix} \right\}$ is defined by

$$\left\{ \begin{matrix} c \\ a,b \end{matrix} \right\} = [a,b,\boldsymbol{g}^{-1}(c)]. \tag{224}$$

Note that $\left\{ \begin{matrix} c \\ a,b \end{matrix} \right\}$ is also a smooth 1-form field since has been defined algebraically from $[a,b,c]$.

For all smooth form fields a, a', b, b', c, c' and smooth scalar field f, the first Christoffel operator satisfies the elementary properties:

$$\begin{aligned} [a+a',b,c] &= [a,b,c] + [a',b,c]. \\ [fa,b,c] &= f[a,b,c]. \\ [a,b+b',c] &= [a,b,c] + [a,b',c]. \\ [a,fb,c] &= f[a,b,c] + (a \cdot \partial f)\boldsymbol{g}(b) \cdot c. \\ [a,b,c+c'] &= [a,b,c] + [a,b,c']. \\ [a,b,fc] &= f[a,b,c]. \end{aligned} \tag{225}$$

Eq.(225) says that $[a,b,c]$ is linear with respect to the first and third arguments, but it is *not* linear with respect to the second argument.

Given the smooth 1-form fields a, b and c, another relevant properties of $[a,b,c]$ are:

$$\begin{aligned} [a,b,c] + [b,a,c] &= a \cdot \partial(\boldsymbol{g}(b) \cdot c) + b \cdot \partial(\boldsymbol{g}(c) \cdot a) - c \cdot \partial(\boldsymbol{g}(a) \cdot b) \\ &\quad + \boldsymbol{g}(b) \cdot [c,a] - \boldsymbol{g}(a) \cdot [b,c] \qquad \text{(226a)} \\ [a,b,c] - [b,a,c] &= \boldsymbol{g}(c) \cdot [a,b]. \qquad \text{(226b)} \\ [a,b,c] + [a,c,b] &= a \cdot \partial(\boldsymbol{g}(b) \cdot c) \qquad \text{(226c)} \\ [a,b,c] - [a,c,b] &= b \cdot \partial(\boldsymbol{g}(c) \cdot a) - c \cdot \partial(\boldsymbol{g}(a) \cdot b) + \boldsymbol{g}(c) \cdot [a,b] \\ &\quad + \boldsymbol{g}(b) \cdot [c,a] - \boldsymbol{g}(a) \cdot [b,c]. \qquad \text{(226d)} \\ [a,b,c] + [c,b,a] &= b \cdot \partial(\boldsymbol{g}(c) \cdot a) + \boldsymbol{g}(c) \cdot [a,b] - \boldsymbol{g}(a) \cdot [b,c] \qquad \text{(226e)} \\ [a,b,c] - [c,b,a] &= a \cdot \partial(\boldsymbol{g}(b) \cdot c) - c \cdot \partial(\boldsymbol{g}(a) \cdot b) + \boldsymbol{g}(b) \cdot [c,a]. \qquad \text{(226f)} \end{aligned}$$

4.5.3 The 2-Extensor field λ

Given the metric structure $(M, \boldsymbol{g})$ and $(U_o, \boldsymbol{g})$, the representative on U_o of its restriction to $\mathcal{U}$, we can construct a connection 2-extensor field λ on U_0, $\lambda \in$

$\sec extU$, such that $a, b \in \sec \bigwedge^1 T^*U$,

$$\lambda : \sec(\bigwedge^1 T^*U \times \bigwedge^1 T^*U) \to \sec \bigwedge^1 T^*U,$$
$$\lambda(a,b) = \partial_c \begin{Bmatrix} c \\ a,b \end{Bmatrix} - a \cdot \partial b. \tag{227}$$

Remark 4.1 In completely analogy to the case of connection 2-extensor field

γ, we also introduce here the extensor fields λ_a and Λ_a. Those object λ and some others constructed from it (see next subsection) enter in the calculations of covariant derivatives of multiform fields and extensor fields defined on U_o. The field λ will be called a *Levi-Civita connection* on U_o.

4.5.4 Riemann-Cartan and Lorentz-Cartan *MGSS's* $(U_o, \boldsymbol{g}, \gamma)$

Consider a triple $(M, \mathbf{g}, \nabla)$ where M is a smooth manifold ($\dim M = n$), $\boldsymbol{g} \in \sec T_2^0 M$ is a Riemannian or a Lorentzian metric tensor and ∇ is an arbitrary *metric compatible connection*. Let $\mathbf{g}$ be the metric extensor corresponding to $\boldsymbol{g}$ and let $\boldsymbol{g}$ the representative of $\mathbf{g}$ on U_o. As we already know the connection ∇ is characterized on U_o by a smooth 2-extensor field γ (and other two extensor fields defined from γ). The metric compatibility means

$$\nabla \boldsymbol{g} = 0, \tag{228}$$

and such condition is obviously expressed on U_0 by

$$\nabla_a^- \boldsymbol{g} = 0. \tag{229}$$

In what follows when $\boldsymbol{g}$ has Euclidean signature ($(n,0)$ or $(0,n)$) we call $(M, \boldsymbol{g}, \nabla)$ a *Riemann-Cartan MCGSS. When* $\mathbf{g}$ has pseudo-euclidean signature $(1, n-1)$ we call $(M, \mathbf{g}, \nabla)$ a *Lorentz-Cartan MCGSS*[38]. We use also such denominations for the triple $(U_o, \boldsymbol{g}, \gamma)$.

4.5.5 Existence Theorem of the $\gamma\boldsymbol{g}$-gauge Rotation Extensor of the *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$

We present now a theorem whose proof may be found in [34] and which pays a crucial role in our theory of the gravitational field in Section 6.

Theorem 4.1 [33] On the *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$ there exists a $(1,2)$-extensor field $\overset{\gamma g}{\omega}(a) = \frac{1}{2}\overset{g}{\mathrm{bif}}[\gamma_a]$, such that for all smooth 1-form fields a and b

$$\gamma_a(b) = \frac{1}{2}\boldsymbol{g}^{-1}(a \cdot \partial \boldsymbol{g})(b) + \overset{\gamma g}{\omega}(a) \underset{g}{\times} b, \tag{230}$$

[38]The *GSS* where $\boldsymbol{g}$ has other possible signatures may be called semi-riemannian.

where

$$\overset{\gamma g}{\omega}(a) \underset{g}{\times} b := \overset{\gamma g}{\omega}(a) \underset{g}{} b - b \underset{g}{} \overset{\gamma g}{\omega}(a). \tag{231}$$

The field $\overset{\gamma g}{\omega}(a)$ is called the *$\gamma\boldsymbol{g}$-gauge rotation field.*

Corollary 4.1 [33] Let $a \mapsto \boldsymbol{g}(a)$ be a metric extensor field defined on U_o and let $a \mapsto \overset{\gamma g}{\omega}(a)$ a smooth $(1,2)$-extensor field defined on U_o. Define a smooth connection 2-extensor field on U_o, say $(a,b) \mapsto \gamma_a(b)$ by

$$\gamma_a(b) = \frac{1}{2}\boldsymbol{g}^{-1}(a \cdot \partial \boldsymbol{g})(b) + \overset{\gamma g}{\omega}(a) \underset{g}{\times} b, \tag{232}$$

for all smooth 1-form fields a and b. Then the triple $(U_o, \boldsymbol{g}, \gamma)$ is a Riemann-Cartan or Lorentz-Cartan *MCGSS* (depending on the signature of $\boldsymbol{g}$).

4.5.6 Some Important Properties of a Metric Compatible Connection

Given the *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$ the operators ∇_a and ∇_a^- satisfy the following properties [33].

(i) *The Ricci Theorem* For all smooth 1-form fields a, b, c we have

$$a \cdot \partial(\boldsymbol{g}(b) \cdot c) = \boldsymbol{g}(\nabla_a b) \cdot c + \boldsymbol{g}(b) \cdot \nabla_a c, \tag{233}$$

Ricci theorem may be generalized for arbitrary smooth multiform fields X and Y, i.e.,

$$a \cdot \partial(\underline{\boldsymbol{g}}(X) \cdot Y) = \underline{\boldsymbol{g}}(\nabla_a X) \cdot Y + \underline{\boldsymbol{g}}(X) \cdot \nabla_a Y. \tag{234}$$

(ii) *Leibniz rule f*or the metric scalar product (i.e., $X \underset{g}{\cdot} Y \equiv \underline{g}(X) \cdot Y$),

$$a \cdot \partial(X \underset{g}{\cdot} Y) = (\nabla_a^- X) \underset{g}{\cdot} Y + X \underset{g}{\cdot} (\nabla_a^- Y), \tag{235}$$

for all smooth multiform fields X and Y.

(iii) For any smooth multiform field X,

$$\nabla_a^- X = \underline{\boldsymbol{g}}(\nabla_a \underline{\boldsymbol{g}}^{-1}(X)). \tag{236}$$

(iv) The action of the operators ∇_a and ∇_a^-, on smooth extensor fields permit us to write the compatibility between the metric and the parallelism as

$$\nabla_a^- \boldsymbol{g} = 0, \tag{237}$$

$$\nabla_a \boldsymbol{g}^{-1} = 0. \tag{238}$$

4.5.7 The Riemann 4-Extensor Field of a *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$

The smooth scalar 4-extensor field on U_o, $(a,b,c,w) \mapsto \overset{\gamma g}{\mathbf{R}}_3(a,b,c,w)$ such that

$$\overset{\gamma g}{\mathbf{R}}_3(a,b,c,w) = -\overset{\gamma g}{\rho}(a,b,c) \cdot \boldsymbol{g}(w), \tag{239}$$

for all smooth 1-from fields a, b, c and w, is called the *Riemann (curvature) 4-extensor field of the structure* $(U_o, \boldsymbol{g}, \gamma)$.

$\overset{\gamma g}{\mathbf{R}}_3$ satisfy the following important properties [34]:

(i) On the parallelism structure (U_o, γ), $\overset{\gamma g}{\mathbf{R}}_3$ is antisymmetric with respect to the first and second variables, i.e.,

$$\overset{\gamma g}{\mathbf{R}}_3(a,b,c,w) = -\overset{\gamma g}{\mathbf{R}}_3(b,a,c,w). \tag{240}$$

(ii) On the parallelism structure (U_o, γ) if the covariant derivative ∇_a is symmetric (i.e., $\nabla_a b - \nabla_b a = [a,b]$), then $\overset{\gamma g}{\mathbf{R}}_3$ possess a cyclic property, i.e.,

$$\overset{\gamma g}{\mathbf{R}}_3(a,b,c,w) + \overset{\gamma g}{\mathbf{R}}_3(b,c,a,w) + \overset{\gamma g}{\mathbf{R}}_3(c,a,b,w) = 0. \tag{241}$$

(iii) On the *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$, $\overset{\gamma g}{\mathbf{R}}_3$ is antisymmetric with respect to the third and fourth variables, i.e.,

$$\overset{\gamma g}{\mathbf{R}}_3(a,b,c,w) = -\overset{\gamma g}{\mathbf{R}}_3(a,b,w,c). \tag{242}$$

(iv) On the *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$, if the covariant derivative ∇_a is symmetric (i.e., $\nabla_a b - \nabla_b a = [a,b]$), then $\overset{\gamma g}{\mathbf{R}}_3$

$$\overset{\gamma g}{\mathbf{R}}_3(a,b,c,w) = \overset{\gamma g}{\mathbf{R}}_3(c,w,a,b). \tag{243}$$

Note that from the Eqs.(240) and (242) we have $\overset{\gamma g}{\mathbf{R}}_3(a,b,c,w) = \overset{\gamma g}{\mathbf{R}}_3(b,a,w,c)$ and under those conditions, given a pair of reciprocal basis $(\{\varepsilon^\mu\}, \{\varepsilon_\mu\})$ on U_o we agree in writing [11, 82]

$$\overset{\gamma g}{\mathbf{R}}_1(\varepsilon_\beta, \varepsilon^\alpha, \varepsilon_\rho, \varepsilon_\sigma) := R_{\beta\ \rho\sigma}^{\ \alpha}, \tag{244}$$

for the *components of the Riemann tensor*. Also, under the same conditions Eq.(219) may be written as

$$\overset{\gamma g}{\mathbf{R}}_2(a,b) = \overset{\gamma g}{\mathbf{R}}_1(\partial_w, a, w, b) = \overset{\gamma g}{\mathbf{R}}_1(a, \partial_w, b, w) \tag{245}$$

and we have

$$R_{\beta\rho} = \overset{\gamma g}{\mathbf{R}}_2(\varepsilon_\beta, \varepsilon_\rho) = \overset{\gamma g}{\mathbf{R}}_1(\varepsilon_\beta, \varepsilon^\alpha, \varepsilon_\rho, \varepsilon_\alpha) = R_{\beta\ \rho\alpha}^{\ \alpha} \tag{246}$$

for the *components of the Ricci tensor*.

The scalar field

$$\overset{\gamma g}{R} = \overset{\gamma g}{\mathbf{R}}_2(\boldsymbol{g}^{-1}(\partial_a), a), \tag{247}$$

i.e., a $\boldsymbol{g}^{-1}$-contraction between the first and second variables of $\overset{\gamma g}{\mathbf{R}}_2$ is called the *Ricci scalar field* (or scalar curvature).

4.5.8 Existence Theorem for the Riemann $(2,2)$-extensor Field on $(U_o, \boldsymbol{g}, \gamma)$

Proposition 4.1 [34] There exists an unique smooth $(2,2)$-extensor field, say $B \mapsto \overset{\gamma g}{\boldsymbol{R}}_2(B)$, such that

$$\overset{\gamma g}{\mathbf{R}}_3(a,b,c,w) = \overset{\gamma g}{\boldsymbol{R}}_2(a \wedge b) \cdot (c \wedge w). \tag{248}$$

This field $B \mapsto \overset{\gamma g}{\boldsymbol{R}}_2(B)$ is called the Riemann $(2,2)$-*extensor field* and we see that Eq.(248) may be though as a factorization of $\overset{\gamma g}{\mathbf{R}}_3$.

Proposition 4.2 The Ricci $(1,1)$-extensor field $b \mapsto \overset{\gamma g}{\boldsymbol{R}}_1(b)$ and the Ricci scalar field R may be written as $\boldsymbol{g}^{-1}$-divergences of the Riemann $(2,2)$-extensor field $B \mapsto \overset{\gamma g}{\boldsymbol{R}}_2(B)$, i.e.,

$$\begin{aligned}\overset{\gamma g}{\boldsymbol{R}}_1(b) &= \boldsymbol{g}^{-1}(\partial_a) \lrcorner \overset{\gamma g}{\boldsymbol{R}}_2(a \wedge b), \\ \overset{\gamma g}{R} &= \boldsymbol{g}^{-1}(\partial_b) \cdot \overset{\gamma g}{\boldsymbol{R}}(b) = \underline{\boldsymbol{g}}^{-1}(\partial_a \wedge \partial_b) \cdot \overset{\gamma g}{\boldsymbol{R}}_2(a \wedge b).\end{aligned} \tag{249}$$

Proof [34] A simple algebraic manipulation of Eq.(216) and Eq. (239) gives the extensor identity

$$\overset{\gamma g}{\mathbf{R}}_1(w,a,b,c) = \overset{\gamma g}{\mathbf{R}}_3(c,b,a,\boldsymbol{g}^{-1}(w)). \tag{250}$$

Now, according to Eq.(219) and Eq. (220), and taking also in account Eq.(250), we have

$$\begin{aligned}a \cdot \overset{g}{\boldsymbol{R}}_1(b) &= \overset{g}{\mathbf{R}}_2(a,b) \\ &= \overset{\gamma g}{\mathbf{R}}_1(\varepsilon^\mu, a, \varepsilon_\mu, b),\ (\mu \text{ summed from 0 to 3}) \\ &= \overset{\gamma g}{\mathbf{R}}_3(b, \varepsilon_\mu, a, \boldsymbol{g}^{-1}(\varepsilon^\mu)),\end{aligned} \tag{251}$$

and utilizing essentially the factorization given by Eq.(248) we have

$$\begin{aligned}a \cdot \overset{\gamma g}{\boldsymbol{R}}_1(b) &= \overset{\gamma g}{\boldsymbol{R}}_2(b \wedge \varepsilon_\mu) \cdot (a \wedge \boldsymbol{g}^{-1}(\varepsilon^\mu)) = (\boldsymbol{g}^{-1}(\varepsilon^\mu) \wedge a) \cdot \overset{\gamma g}{\boldsymbol{R}}_2(\varepsilon_\mu \wedge b) \\ &= a \cdot (\boldsymbol{g}^{-1}(\varepsilon^\mu) \lrcorner \overset{g}{\boldsymbol{R}}_2(\varepsilon_\mu \wedge b)),\end{aligned}$$

i.e., $\overset{\gamma g}{\boldsymbol{R}}_1(b) = \boldsymbol{g}^{-1}(\varepsilon^\mu) \lrcorner \overset{\gamma g}{\boldsymbol{R}}_2(\varepsilon_\mu \wedge b) = \boldsymbol{g}^{-1}(\partial_a) \lrcorner \overset{\gamma g}{\boldsymbol{R}}_2(a \wedge b)$. ■

4.5.9 The Einstein $(1,1)$-Extensor Field

The smooth $(1,1)$-extensor field, $a \mapsto \overset{\gamma g}{\boldsymbol{G}}(a)$, given by

$$\overset{\gamma g}{\boldsymbol{G}}(a) := \overset{\gamma g}{\boldsymbol{R}}_1(a) - \frac{1}{2}\boldsymbol{g}(a)\overset{\gamma g}{R}, \tag{252}$$

is called the *Einstein* $(1,1)$-*extensor field.*

4.6 Riemann and Lorentz *MCGSS's* $(U_o, \boldsymbol{g}, \lambda)$

4.6.1 Levi-Civita Covariant Derivative

It is important to have in mind that the unique *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$ where the covariant derivative ∇_a is symmetric is the one where the connection 2-extensor field γ is precisely the Levi-Civita one, that we already denoted by λ (recall Eq.(227)). In this case the *MCGSS* $(M, \boldsymbol{g}, \nabla = D)$ is said a Riemann or *Lorentz MCGSS* depending on the signature of $\boldsymbol{g}$. We also use this denomination for the structure $(U_o, \boldsymbol{g}, \lambda)$. The covariant derivatives defined by λ are precisely the Levi-Civita covariant derivatives D_a and D_a^-. Some properties of the curvature extensors of the *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$ which are important for this paper will be given below, and for that important case *simplified* notations will be used.

The covariant derivative operators associated to the connection 2-extensor field λ will be denoted D_a and D_a^-, and

$$D_a X := a \cdot \partial X + \Lambda_a(X), \tag{253a}$$

$$D_a^- X := a \cdot \partial X - \Lambda_a^\dagger(X), \tag{253b}$$

for all smooth multiform field $X \in \sec \bigwedge T^*U$ and are called *Levi-Civita covariant derivative operators.* Note that Λ_a is the generalized of λ_a and $\Lambda_a^\dagger$ is the adjoint of Λ_a.

4.6.2 Properties of D_a

(i) For all smooth 1-form fields a, b and c it is:

$$(D_a b) \cdot c = \left\{ \begin{matrix} c \\ a, b \end{matrix} \right\}. \tag{254}$$

Indeed, using Eq.(253a) and Eq.(227) we have immediately

$$D_a b = a \cdot \partial b + \lambda_a(b) = a \cdot \partial b + \partial_c \left\{ \begin{matrix} c \\ a, b \end{matrix} \right\} - a \cdot \partial b = \partial_c \left\{ \begin{matrix} c \\ a, b \end{matrix} \right\}.$$

Now, utilizing the formula for multiform differentiation $c \cdot \partial_n \phi(n) = c \cdot (\partial_n \phi(n))$, with ϕ a scalar function of 1-form variable, the formula $c \cdot \partial_n f(n) = f(c)$, where f is a multiform function of 1-form variable and taking into account the linearity of the Christoffel operator with respect to the superior argument, we have:

$$(D_a b) \cdot c = c \cdot (\partial_n \left\{ \begin{matrix} n \\ a, b \end{matrix} \right\}) = c \cdot \partial_n \left\{ \begin{matrix} n \\ a, b \end{matrix} \right\} = \left\{ \begin{matrix} c \\ a, b \end{matrix} \right\}.$$

(ii) In order to get acquainted with the algebraic manipulations of the multiform and extensor calculus we give the details in the derivation of the *Ricci theorem for the* Levi-Civita covariant derivative D_a, i.e.,

$$a \cdot \partial(\boldsymbol{g}(b) \cdot c) = \boldsymbol{g}(D_a b) \cdot c + \boldsymbol{g}(b) \cdot D_a c, \tag{255}$$

where $a, b, c \in \sec \bigwedge^1 T^*U$ are smooth 1-form fields.

Utilizing Eq.(254), Eq.(222) and Eq.(226c), we have

$$
\begin{aligned}
\boldsymbol{g}(D_a b) \cdot c + \boldsymbol{g}(b) \cdot D_a c &= (D_a b) \cdot \boldsymbol{g}(c) + D_a c \cdot \boldsymbol{g}(b) \\
&= \begin{Bmatrix} \boldsymbol{g}(c) \\ a, b \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{g}(b) \\ a, c \end{Bmatrix} \\
&= [a, b, \boldsymbol{g}^{-1}(\boldsymbol{g}(c))] + [a, c, \boldsymbol{g}^{-1}(\boldsymbol{g}(b))] \\
&= [a, b, c] + [a, c, b] = a \cdot \partial(\boldsymbol{g}(b) \cdot c).
\end{aligned}
$$

(iii) This theorem is also valid for smooth multiform fields X, Y, i.e.,

$$
a \cdot \partial(X \cdot Y) = \underline{\boldsymbol{g}}(D_a X) \cdot Y + \underline{\boldsymbol{g}}(X) \cdot D_a Y, \tag{256}
$$

(iv) The Levi-Civita covariant derivative D_a is symmetric, i.e., ,

$$
D_a b - D_b a = [a, b], \tag{257}
$$

for all smooth 1-form fields a and b.

Take three smooth 1-form fields a, b and c. Utilizing Eq.(254), Eq.(224) and Eq.(226b), we can write

$$
\begin{aligned}
(D_a b) \cdot c - (D_b a) \cdot c &= \begin{Bmatrix} c \\ a, b \end{Bmatrix} - \begin{Bmatrix} c \\ b, a \end{Bmatrix} = [a, b, \boldsymbol{g}^{-1}(c)] - [b, a, \boldsymbol{g}^{-1}(c)] \\
&= \boldsymbol{g}(\boldsymbol{g}^{-1}(c)) \cdot [a, b] = [a, b] \cdot c,
\end{aligned}
$$

which from the non degeneracy of the scalar product gives

$$
D_a b - D_b a = [a, b]. \blacksquare
$$

(v) The Levi-Civita covariant derivative D_a is $\boldsymbol{g}$*-compatible,* i.e.,

$$
D_a^- \boldsymbol{g} = 0. \tag{258}
$$

Take three smooth 1-form fields a, b and c. Utilizing Eq.(209) and the Ricci theorem (Eq.(255)), we have

$$
\begin{aligned}
(D_a^- \boldsymbol{g})(b) \cdot c &= a \cdot \partial(\boldsymbol{g}(b) \cdot c) - \boldsymbol{g}(D_a b) \cdot c - \boldsymbol{g}(b) \cdot D_a c, \\
&= 0,
\end{aligned}
$$

which implies that $(D_a^- \boldsymbol{g})(b) = 0$, i.e., $D_a^- \boldsymbol{g} = 0$.

We emphasize that the symmetry property and the $\boldsymbol{g}$-compatibility uniquely characterizes the Levi-Civita covariant derivative, i.e., there exists an unique pair of covariant derivative operators ∇_a and ∇_a^- such that ∇_a is symmetric (i.e., $\nabla_a b - \nabla_b a = [a, b]$) and $\nabla_a^- \boldsymbol{g} = 0$ (i.e., ∇_a^- is $\boldsymbol{g}$-compatible). Those ∇_a and ∇_a^- are precisely D_a and D_a^-.

Properties of $\boldsymbol{R}_2(B)$ and $\boldsymbol{R}_1(b)$ On a Riemann or Lorentz *MCGSS* $(U_o, \boldsymbol{g}, \lambda)$ we use simplified notations, the Riemann $(2,2)$- extensor field is denoted $B \mapsto \boldsymbol{R}_2(B)$ and the Ricci $(1,1)$-extensor is denoted $b \mapsto \boldsymbol{R}_1(b)$. Those objects now have three additional properties besides the ones those objects have on a general *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$. We have

(i) $\boldsymbol{R}_2(B)$ is adjoint symmetric, i.e.,

$$\boldsymbol{R}_2(B) = \boldsymbol{R}_2^{\dagger}(B). \tag{259}$$

(ii) $\boldsymbol{R}_1(b)$ is adjoint symmetric, i.e.,

$$\boldsymbol{R}_1(b) = \boldsymbol{R}_1^{\dagger}(b). \tag{260}$$

(iii) The *matrix element* for the Ricci extensor field is given by the notable formula

$$\boldsymbol{R}_1(b) \cdot c = \partial_a \cdot \rho(a, b, c), \tag{261}$$

i.e., the divergent of the curvature operator with respect to the first variable.

4.6.3 Levi-Civita Differential Operators

On the Riemann or Lorentz *GSS* $(U_o, \boldsymbol{g}, \lambda)$ we can introduce three differential operators : the gradient $\underset{g}{\boldsymbol{D}}$, the divergent $\underset{g}{\boldsymbol{D}\lrcorner}$ and the *rotational* $\boldsymbol{D}\wedge$ acting on smooth multiform fields.

- The *gradient of a smooth multiform field* $X \in \sec \bigwedge T^*U$ is defined by

$$\underset{g}{\boldsymbol{D}} X := \partial_a \underset{g}{} (D_a^- X), \tag{262}$$

  i.e., $\underset{g}{\boldsymbol{D}} X = \varepsilon^{\mu} \underset{g}{} (D_{\varepsilon_\mu}^- X) = \varepsilon_{\mu} \underset{g}{} (D_{\varepsilon^\mu}^- X)$, where $(\{\varepsilon^\mu\}, \{\varepsilon_\mu\})$ is a pair of arbitrary reciprocal basis defined on U_o.

- The *divergent of a smooth multiform field* $X \in \sec \bigwedge T^*U$ is defined by

$$\underset{g}{\boldsymbol{D}\lrcorner} X = \partial_a \underset{g}{\lrcorner} (D_a^- X). \tag{263}$$

- The *rotacional of a smooth multiform field* $X \in \sec \bigwedge T^*U$ is defined by

$$\boldsymbol{D} \wedge X = \partial_a \wedge (D_a^- X). \tag{264}$$

The main properties of those operators are:

(i) For any smooth multiform field $X \in \sec \bigwedge T^*U$,

$$\underset{g}{\boldsymbol{D}} X = \underset{g}{\boldsymbol{D}\lrcorner} X + \boldsymbol{D} \wedge X, \tag{265}$$

i.e., $\underset{g}{\boldsymbol{D}} = \underset{g}{\boldsymbol{D}\lrcorner} + \boldsymbol{D}\wedge$.

(ii) For any smooth multiform field $X \in \sec \bigwedge T^*U$

$$\underset{g}{\boldsymbol{D}\lrcorner} X = \frac{1}{\sqrt{|\det[\boldsymbol{g}]|}} \underline{g}(\partial \lrcorner \sqrt{|\det[\boldsymbol{g}]|} \underline{g}^{-1}(X)), \tag{266}$$

$$\boldsymbol{D} \wedge X = \partial \wedge X. \tag{267}$$

4.7 Deformation of *MCGSS* Structures

4.7.1 Enter the Plastic Distortion Field h

Minkowski Metric on U_o Let M be a smooth manifold with $\dim M = 4$. Consider the canonical spaces $\mathbf{U}$ and U defined by local coordinates $(\mathcal{U}, \phi)_o$, and the canonical dual bases[39] $\{\mathbf{b}_\mu\}$ and $\{\boldsymbol{\beta}^\mu\}$ for $\mathbf{U}$ and U, $\boldsymbol{\beta}^\mu(\mathbf{b}_\nu) = \delta^\mu_\nu$. Moreover, we denote the reciprocal basis of the basis $\{\boldsymbol{\beta}^\mu\}$ by $\{\beta_\mu\}$, with $\boldsymbol{\beta}^\mu \cdot \beta_\nu = \delta^\mu_\nu$. Note that due to the identifications that define the canonical space, we can also write that $\boldsymbol{\beta}^\mu, \beta_\nu \in \sec \bigwedge^1 T^*U$ for all $x \in U_0$.

Recalling Section 2.8.4 we introduce the smooth extensor field on U_0, say $\eta \in \sec(1,1)\text{-}extU$ defined for any $a \in \sec \bigwedge^1 T^*U$ by $a \mapsto \eta(a) \in \sec \bigwedge^1 T^*U$, such that

$$\eta(a) = \boldsymbol{\beta}^0 a \boldsymbol{\beta}^0. \tag{268}$$

The field η is a well defined metric extensor field on U_0 (i.e., η is symmetric and non degenerated) and is called *Minkowski metric extensor*[40].

The main properties of η are:

(i) η is a constant $(1,1)$-extensor field, i.e., $a \cdot \partial \eta = 0$.

(ii) The 1-forms of the canonical basis $\{\boldsymbol{\beta}^\mu\}$ are η orthonormal (i.e., $\eta(\boldsymbol{\beta}^\mu) \cdot \boldsymbol{\beta}^\nu = \delta^{\mu\nu}$); $\boldsymbol{\beta}^0$ is an eigenvector with eigenvalue 1, i.e., $\eta(\boldsymbol{\beta}^0) = \boldsymbol{\beta}^0$, and $\boldsymbol{\beta}^k$ $(k = 1,2,3)$ are eigenvectors with eigenvalues -1, i.e., $\eta(\boldsymbol{\beta}^k) = -\boldsymbol{\beta}^k$.

Then, η has signature $(1,3)$.

(iii) $\mathrm{tr}[\eta] = -2$ and $\det[\eta] = -1$.

(iv) The extended of η has the same generator of η, i.e., for any $X \in \sec \bigwedge^1 T^*U$, $\underline{\eta}(X) = \boldsymbol{\beta}^0 X \boldsymbol{\beta}^0$.

(v) η is orthogonal canonical, i.e., $\eta = \eta^*$ (recall that $\eta^* = (\eta^\dagger)^{-1} = (\eta^{-1})^\dagger$). Then, $\eta^2 = i_d$.

4.7.2 Lorentzian Metric

Given a metric tensor $\boldsymbol{g} \in \sec T^0_2 M$ of signature $(1,3)$, if the $(1,1)$-extensor field associated to $\boldsymbol{g}$ is $\mathbf{g}$ and its representative on U_0 is $\boldsymbol{g}$, then $\boldsymbol{g}$ also has the same signature as the Minkowski metric extensor, i.e., signature$(\boldsymbol{g})$ = signature(η) =

[39] For spacetime structures, as usual the Greek indices take the values $0,1,2,3$ and Latin indices take the values $1,2,3$.

[40] We eventually call η the Minkowski metric when no confusion arises.

$(1,3)$. In what follows $\mathbf{g}$ (and its representative $\boldsymbol{g}$) will be called a *Lorentzian metric extensor field*, or when non confusion arises, simply Lorentzian metric.

Recalling Section 2.8.4 we know that there exists a (non unique) $(1,1)$-extensor field $\boldsymbol{h}$ given by:

$$\boldsymbol{h}(a) = \sum_{\mu=0}^{3} \sqrt{|\lambda^\mu|}(a \cdot v^\mu)\boldsymbol{\beta}^\mu, \tag{269}$$

where the $\lambda^\mu \in \sec \bigwedge^0 T^*U$ are the eigenvalue fields of $\boldsymbol{g}$ and $v^\mu \in \sec \bigwedge^1 T^*U$ are the associated eigenvector fields of $\boldsymbol{g}$ such that

$$\boldsymbol{g} = \boldsymbol{h}^\dagger \eta \boldsymbol{h}. \tag{270}$$

The 1-form $\{v^\mu\}$ defines an *euclidean orthonormal basis* for V, i.e., $v^\mu \cdot v^\nu = \delta^{\mu\nu}$. As we know $\boldsymbol{h}$ is not unique and is defined modulo a local Lorentz transformation, i.e., $\boldsymbol{h}$ and $\boldsymbol{h}' = \Lambda \boldsymbol{h}$ determine the same $\boldsymbol{g}$, if $\Lambda \in \sec(1,1)$-$extU$ is a local Lorentz transformation, i.e., for any $a, b \in \sec \bigwedge^1 T^*U$, $\Lambda(a) \underset{\eta}{\cdot} \Lambda(b) = a \underset{\eta}{\cdot} b$. In what follows we call $\boldsymbol{h}$ a *plastic gauge distortion field* on U_o.

4.7.3 On Elastic and Plastic Deformations

The wording gauge is of course, well justified, since $\boldsymbol{h}$ can only be determined modulus a local Lorentz transformation. The wording distortion is also well justified in view of Eq.(270), $\boldsymbol{h}$ distorts η into $\boldsymbol{g}$. The wording plastic is justified as follows. In the theory of deformations and defects on continuum media [108] it is introduced two different kinds of deformations for any medium that lives on a manifold M carrying a metric field, say $\mathring{\mathbf{g}}$. Those deformations are: (i) an *elastic* one where the deformation of the medium is described by diffeomorphism $\mathtt{h}$: $M \to M$ which induces a *deformed* metric given by the pullback metric[41] $\mathbf{g} = \mathtt{h}^* \mathring{\mathbf{g}}$ on M and which is used in the definition of the *Cauchy-Green tensor* [36], and (ii): a *plastic* one where the deformation of the medium is such that the new effective metric $\mathbf{g}$ on M defining distance measurements *cannot* be described by the pullback of any diffeomorphism as in the elastic case. In case (i) we can show that the pullback $D' = \mathtt{h}^* D$ of the Levi-Civita connection D of $\mathring{\mathbf{g}}$, has the same torsion and Riemann curvature tensors then D, i.e., they are null if D has null curvature and torsion tensors[42], whereas in case (ii) the medium can conveniently be described by a new effective *MCGSS* with a connection ∇ that is $\boldsymbol{g}$ compatible ($\nabla \boldsymbol{g} = 0$) but that has in general non null torsion and Riemann curvature tensors.

[41] Take notice that here $\mathtt{h}^*$ denotes the pulback mapping.

[42] This result, which can be easily proven is shown, e.g, in Remark 250 in [82]. We take the opportunity to call the reader's attention that in [68] it is presented a theory for the gravitational field where the field $\boldsymbol{g}$ has been identified as a result of an elastic deformation. Such identification is, of course, incorrect. The arXiv version of the paper contains a correction.

Now, although there exists a distortion field $\boldsymbol{h}$ corresponding to the pullback $\mathtt{h}^*$ of any diffeomorphism [34], there are distortion fields $\boldsymbol{h}$ to which there corresponds *no* diffeomorphism. Thus the wording *plastic* is justified, even more because we are going to see below that $\boldsymbol{h}$ also deforms in a precise *sense* the Levi-Civita connection of $\overset{\circ}{\boldsymbol{g}}$ and thus produces a connection on M that is not compatible with $\overset{\circ}{\boldsymbol{g}}$. And in this sense we can also say that $\boldsymbol{h}$ distorts the parallelism structure defined by the Levi-Civita connection of $\overset{\circ}{\boldsymbol{g}}$.

4.7.4 Construction of a Lorentzian Metric Field on U_o

Proposition 4.1 [33] Let $\boldsymbol{h}$ be a $(1,1)$-extensor field on U_0, defined by

$$\boldsymbol{h}(a) = \sum_{\mu=0}^{3} \rho^\mu \Lambda(a) \cdot \beta^\mu \beta^\mu, \tag{271}$$

where $\rho^\mu \in \sec \bigwedge T^*U$ are positive scalar fields on U_o (i.e., $\rho^\mu(x) > 0$, for $x \in U_0$) and Λ is an orthogonal $(1,1)$-extensor field on U_0 (i.e., $\Lambda = \Lambda^\diamond$; then the $(1,1)$-extensor field $\boldsymbol{g}$ on U_o given by

$$\boldsymbol{g} = \boldsymbol{h}^\dagger \eta \boldsymbol{h}, \tag{272}$$

is a Lorentzian metric extensor field (i.e., signature$(\boldsymbol{g}) = (1,3)$).

The non null scalar fields $(\rho^0)^2$ and $-(\rho^1)^2, -(\rho^2)^2, -(\rho^3)^2$ are the eigenvalue fields of $\boldsymbol{g}$, and the non null 1-form fields $\Lambda^\dagger(\boldsymbol{\beta}^0)$ and $\Lambda^\dagger(\boldsymbol{\beta}^1), \boldsymbol{\beta}^\dagger(\beta^2), \boldsymbol{\beta}^\dagger(\beta^3)$ are the associated eigenvector fields of $\boldsymbol{g}$.

4.8 Deformation of a Minkowski-Cartan $(U_o, \eta, \varkappa)$ *MCGSS* into a Lorentz-Cartan *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$

Let M be 4-dimensional (with enough structure to be part of a Lorentzian spacetime structure as defined in Section 1). Let $(\mathcal{U}, \phi)_0$ a chart for $\mathcal{U} \subset M$ containing $0 \in \mathcal{U}$ and let $U_o \subset U$ be defined as previously. A *MCGSS* $(U_o, \eta, \varkappa)$ where η is the Minkowski metric extensor on U_o and $\varkappa$ is a connection extensor field on U_o compatible with η is said a *Minkowski-Cartan MCGSS*.

The metric compatibility may be written $\overset{\varkappa\eta^-}{\mathcal{D}_a} \eta = 0$, where $\overset{\varkappa\eta^-}{\mathcal{D}_a}$ is one of the covariant derivative operators associated to $\varkappa$. Before going on, a crucial observation is needed.

Remark 4.3 Note that we did not impose that the manifold M carries a Minkowski metric field $\boldsymbol{\eta}$ as *given* by Eq.(6) for if that was the case we would have $M \simeq \mathbb{R}^4$, which is not being considered at this moment.

Now, let us recall that the Theorem 4.1 (Eq.(230)) implies the existence of a *$\varkappa\eta$-gauge rotation field*, say $\overset{\varkappa\eta}{\Omega}$ such

$$\varkappa_a(b) = \overset{\varkappa\eta}{\Omega}(a) \underset{\eta}{\times} b = \overset{\varkappa\eta}{\Omega}(a) \times \eta(b). \tag{273}$$

Also, according to Eq.(192a), the covariant derivative $\overset{\varkappa\eta}{\mathcal{D}}_a$ when acting on a smooth 1-form field is given by

$$\overset{\varkappa\eta}{\mathcal{D}}_a b = a \cdot \partial b + \overset{\varkappa\eta}{\Omega}(a) \underset{\eta}{\times} b. \tag{274}$$

Remark 4.4 Keep in mind that in a general Minkowski-Cartan *GSS* $(U_o, \eta, \varkappa)$, $\overset{\varkappa\eta}{\mathcal{D}}_a$ is not the representative of the Levi-Civita connection of $\boldsymbol{\eta}$ and indeed, in general possess non null torsion and Riemann curvature tensors.

We recall that a *MCGSS* $(M, \boldsymbol{g}, \nabla)$ where M is a 4-dimensional manifold as in the previous subsection, $\boldsymbol{g}$ is a Lorentzian metric ∇ is an arbitrary metric compatible connection on M, i.e., $\nabla \boldsymbol{g} = 0$, is said to be a *Lorentz-Cartan MCGSS*. Let U_o, $\boldsymbol{g}$, and γ as defined previously. We recall that we agreed in calling also $(U_o, \boldsymbol{g}, \gamma)$ a Lorentz-Cartan *MCGSS*.

The compatibility between $\boldsymbol{g}$ and γ may be written as $\overset{\gamma g}{\nabla}{}^{-}_{a} \boldsymbol{g} = 0$, where $\overset{\gamma g}{\nabla}{}^{-}_{a}$ is one of the covariant derivative operators associated to γ.

The Theorem 4.1 implies in this case the existence of a $\gamma\boldsymbol{g}$-gauge rotation field $\overset{\gamma g}{\omega}(a)$ such that

$$\gamma_a(b) = \frac{1}{2}\boldsymbol{g}^{-1}(a \cdot \partial \boldsymbol{g})(b) + \overset{\gamma g}{\omega}(a) \underset{g}{\times} b = \frac{1}{2}\boldsymbol{g}^{-1}(a \cdot \partial \boldsymbol{g})(b) + \overset{\gamma g}{\omega}(a) \times \boldsymbol{g}(b). \tag{275}$$

According to Eq.(192a), the covariant derivative $\overset{\gamma g}{\nabla}_a$, when acting on a smooth 1-form field is given by

$$\overset{\gamma g}{\nabla}_a b = a \cdot \partial b + \frac{1}{2}\boldsymbol{g}^{-1}(a \cdot \partial \boldsymbol{g})(b) + \overset{\gamma g}{\omega}(a) \underset{g}{\times} b. \tag{276}$$

4.8.1 *h*-Distortions of Covariant Derivatives

Theorem 4.2. The operators $\overset{\gamma g}{\nabla}_a$ and $\overset{\gamma g}{\nabla}{}^{-}_{a}$ of the Lorentz-Cartan *GSS* are related with the operators $\overset{\varkappa\eta}{\mathcal{D}}_a$ and $\overset{\varkappa\eta}{\mathcal{D}}{}^{-}_{a}$ of the Minkowski-Cartan *GSS* by

$$\overset{\gamma g}{\nabla}_a b = \boldsymbol{h}^{-1}(\overset{\varkappa\eta}{\mathcal{D}}_a \boldsymbol{h}(b)), \tag{277}$$

$$\overset{\gamma g}{\nabla}{}^{-}_{a} b = \boldsymbol{h}^{\dagger}(\overset{\varkappa\eta}{\mathcal{D}}{}^{-}_{a} \boldsymbol{h}^{\clubsuit}(b)), \tag{278}$$

where $\boldsymbol{h}$ is the *plastic distortion field* introduce above such that $\boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}$.

In order to prove the Theorem 4.2 we shall need the results of the following two lemmas.

Lemma 4.1 Let $(\overset{1}{\nabla}_a, \overset{1}{\nabla}{}^{-}_{a})$ be any pair of associated covariant derivative operators, i.e., $\overset{1}{\nabla}_a$ and $\overset{1}{\nabla}{}^{-}_{a}$ satisfy Eq.(198),

$$a \cdot \partial(b \cdot c) = (\overset{1}{\nabla}_a b) \cdot c + b \cdot \overset{1}{\nabla}{}^{-}_{a} c. \tag{279}$$

Then the pair of covariant derivative operators $(\overset{2}{\nabla}, \overset{2}{\nabla}{}_a^-)$, defined by

$$\overset{2}{\nabla}_a b = \boldsymbol{h}^{-1}(\overset{1}{\nabla}_a \boldsymbol{h}(b)), \tag{280}$$

$$\overset{2}{\nabla}{}_a^- b = \boldsymbol{h}^{\dagger}(\overset{1}{\nabla}{}_a^- \boldsymbol{h}^{\clubsuit}(b)), \tag{281}$$

where $\boldsymbol{h}$ is a smooth invertible $(1,1)$-extensor field is also a pair of associated covariant derivative operators, i.e., $\overset{2}{\nabla}_a$ and $\overset{2}{\nabla}{}_a^-$ satisfy Eq.(198),

$$a \cdot \partial(b \cdot c) = (\overset{2}{\nabla}_a b) \cdot c + b \cdot (\overset{2}{\nabla}{}_a^- c). \tag{282}$$

Proof Given that $\overset{1}{\nabla}_a$ and $\overset{1}{\nabla}{}_a^-$ are well defined covariant derivative operators they satisfy Eqs.(193), (194) and (195). Then a simple algebraic manipulation shows that $\overset{2}{\nabla}_a$ and $\overset{2}{\nabla}{}_a^-$ also satisfy those same equations. Thus, $\overset{2}{\nabla}_a$ and $\overset{2}{\nabla}{}_a^-$ are also well defined covariant derivative operators.

On the other side, using Eq.(198) we have

$$\overset{1}{\nabla}_a \boldsymbol{h}(b) \cdot \boldsymbol{h}^{\clubsuit}(c) + \boldsymbol{h}(b) \cdot \overset{1}{\nabla}{}_a^- \boldsymbol{h}^{\clubsuit}(c) = a \cdot \partial(\boldsymbol{h}(b) \cdot \boldsymbol{h}^{\clubsuit}(c)),$$

and after some algebraic manipulations, using Eqs.(280) and (281) we get

$$\boldsymbol{h}^{-1}(\overset{1}{\nabla}_a \boldsymbol{h}(b)) \cdot c + b \cdot \boldsymbol{h}^{\dagger}(\overset{1}{\nabla}{}_a^- \boldsymbol{h}^{\clubsuit}(c)) = a \cdot \partial(\boldsymbol{h}^{-1}\boldsymbol{h}(b) \cdot c)$$
$$(\overset{2}{\nabla}_a b) \cdot c + b \cdot \overset{2}{\nabla}{}_a^- c = a \cdot \partial(b \cdot c).$$

We see then that $\overset{2}{\nabla}_a$ and $\overset{2}{\nabla}{}_a^-$ also satisfy Eq.(198), i.e., $(\overset{2}{\nabla}_a, \overset{2}{\nabla}{}_a^-)$ is a pair of associated covariant derivative operators. ■

Lemma 4.2 Let $\boldsymbol{h}$ be a plastic distortion field such that $\boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}$, and let $\overset{1}{\nabla}{}_a^-$ and $\overset{2}{\nabla}{}_a^-$ be a pair of associated covariant derivative operators acting on the module of the smooth (p,q)-extensor fields. Then

$$\overset{2}{\nabla}{}_a^- \boldsymbol{g} = \boldsymbol{h}^{\dagger}(\overset{1}{\nabla}{}_a^- \eta)\boldsymbol{h}. \tag{283}$$

Proof Indeed, according to the Eq.(199b),utilizing Eqs.(280) and (280) and also taking into account the Lemma 4.1, we have

$$(\overset{2}{\nabla}{}_a^- \boldsymbol{g})(b) = \overset{2}{\nabla}{}_a^- \boldsymbol{g}(b) - \boldsymbol{g}(\overset{2}{\nabla}_a b) = \boldsymbol{h}^{\dagger}\overset{1}{\nabla}{}_a^- \boldsymbol{h}^{\clubsuit}\boldsymbol{h}^{\dagger}\eta\boldsymbol{h}(b) - \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}\boldsymbol{h}^{-1}\overset{1}{\nabla}_a \boldsymbol{h}(b)$$
$$= \boldsymbol{h}^{\dagger}(\overset{1}{\nabla}{}_a^- \eta\boldsymbol{h}(b)) - \eta\overset{1}{\nabla}_a \boldsymbol{h}(b)) = \boldsymbol{h}^{\dagger}(\overset{1}{\nabla}{}_a^- \eta)\boldsymbol{h}(b),$$

i.e., $\overset{2}{\nabla}{}_a^- \boldsymbol{g} = \boldsymbol{h}^{\dagger}(\overset{1}{\nabla}{}_a^- \eta)\boldsymbol{h}$. ■

Eq.(283) shows that if $\overset{1}{\nabla}{}^{-}_{a}$ is η-compatible (i.e., $\overset{1}{\nabla}{}^{-}_{a}\eta = 0$) iff $\overset{2}{\nabla}{}^{-}_{a}$ is $\boldsymbol{g}$-compatible (i.e., $\overset{2}{\nabla}{}^{-}_{a}\boldsymbol{g} = 0$).

Or in other worlds, the operators $\overset{1}{\nabla}_{a}$ and $\overset{1}{\nabla}{}^{-}_{a}$ are covariant derivative operators on the Minkowski-Cartan *MCGSS* iff $\overset{2}{\nabla}_{a}$ and $\overset{2}{\nabla}{}^{-}_{a}$ are covariant derivative operators on the Lorentz-Cartan *MCGSS*.

Proof of Theorem 4.2 The theorem follows at once from Lemmas 4.1. and 4.2 using the identifications:

$$\overset{\varkappa\eta}{\mathcal{D}}_{a} \equiv \overset{1}{\nabla}_{a},\ \overset{\varkappa\eta-}{\mathcal{D}}_{a} \equiv \overset{1}{\nabla}{}^{-}_{a} \text{ and } \overset{\gamma g}{\nabla}_{a} \equiv \overset{2}{\nabla}_{a}, \overset{\gamma g}{\nabla}{}^{-}_{a} \equiv \overset{2}{\nabla}{}^{-}_{a}. \blacksquare$$

4.8.2 Existence of a 2-Extensor Coupling Field Between the Minkowski-Cartan *MCGSS* and Lorentz-Cartan *MCGSS*

Proposition 4.3 There exists a smooth 2-extensor field on U_0, such that for all $a \in \sec\bigwedge^{1}T^{*}U$, there exists $f_a : \sec\bigwedge^{1}T^{*}U \to \sec\bigwedge^{1}T^{*}U$, $b \mapsto f_a(b)$, which is η-adjoint antisymmetric [43] (i.e., $f_a = -f_a^{\dagger(\eta)}$) such that

$$\underline{\boldsymbol{h}}\overset{\gamma g}{\omega}(a) = \overset{\varkappa\eta}{\Omega}(a) + \frac{1}{2}\overset{\eta}{\text{bif}}[f_a]. \tag{284}$$

Such a 2-extensor field is given by the formula

$$f_a(b) = (a\cdot\partial\boldsymbol{h})\boldsymbol{h}^{-1}(b) - \frac{1}{2}\eta\boldsymbol{h}^{\clubsuit}(a\cdot\partial\boldsymbol{g})\boldsymbol{h}^{-1}(b). \tag{285}$$

The proof is a simple calculation.

4.8.3 The Gauge Riemann and Ricci Fields

Let $(a,b,c) \mapsto \overset{\varkappa\eta}{\rho}(a,b,c)$ and $(a,b,c) \mapsto \overset{\gamma g}{\rho}(a,b,c)$ be the curvature operators of the *MCGSS*'s $(U_o,\eta,\varkappa)$ and $(U_o,\boldsymbol{g},\gamma)$ as previously defined and $(a,b,c,w) \mapsto \overset{\varkappa\eta}{\mathbf{R}}_3(a,b,c,w)$ and $(a,b,c,w) \mapsto \overset{\gamma g}{\mathbf{R}}_3(a,b,c,w)$ the corresponding Riemann 4-extensor fields.

Let $B \mapsto \overset{\varkappa\eta}{\mathbf{R}}_2(B)$ and $B \mapsto \overset{\gamma g}{\mathbf{R}}_2(B)$ be the Riemann 2-extensor fields of $(M,\eta,\varkappa)$ and $(M,\boldsymbol{g},\gamma)$.

We related $\overset{\varkappa\eta}{\rho}(a,b,c)$ with $\overset{\gamma g}{\rho}(a,b,c)$, utilizing Eq.(277). Indeed,

$$\begin{aligned}\overset{\gamma g}{\rho}(a,b,c) &= [\nabla_a,\nabla_b]c - \nabla_{[a,b]}c \\ &= \boldsymbol{h}^{-1}([\mathcal{D}_a,\mathcal{D}_b]\boldsymbol{h}(c) - \mathcal{D}_{[a,b]}\boldsymbol{h}(c)) \\ \overset{\gamma g}{\rho}(a,b,c) &= \boldsymbol{h}^{-1}(\overset{\varkappa}{\rho}(a,b,\boldsymbol{h}(c))).\end{aligned} \tag{286}$$

[43] The $\boldsymbol{g}$-adjoint (or metric adjoint) of a $(1,1)$-extensor t is the $(1,1)$-extensor $t^{\dagger(g)} \equiv \boldsymbol{g}\circ t^{\dagger}\circ\boldsymbol{g}^{-1}$.

We relate $\overset{\varkappa\eta}{\mathbf{R}}_3(a,b,c,w)$ with $\overset{\gamma_g}{\mathbf{R}}_3(a,b,c,w)$, utilizing Eq.(286). Indeed,

$$\begin{aligned}
\overset{\gamma_g}{\mathbf{R}}_3(a,b,c,w) &= -\overset{\gamma_g}{\rho}(a,b,c)\cdot\boldsymbol{g}(w) \\
&= -\boldsymbol{h}^{-1}(\overset{\varkappa\eta}{\rho}(a,b,\boldsymbol{h}(c)))\cdot\boldsymbol{h}^{\dagger}\eta\boldsymbol{h}(w) \\
&= -\overset{\varkappa\eta}{\rho}(a,b,\boldsymbol{h}(c))\cdot\eta\boldsymbol{h}(w) \\
\overset{\gamma_g}{\mathbf{R}}_3(a,b,c,w) &= \overset{\varkappa\eta}{\mathbf{R}}_3(a,b,\boldsymbol{h}(c),\boldsymbol{h}(w)). \qquad (287)
\end{aligned}$$

We related $\overset{\varkappa\eta}{\boldsymbol{R}}_2(B)$ with $\overset{\gamma_g}{\boldsymbol{R}}_2(B)$, utilizing the factorization of $\mathbf{R}_3$, and the Eqs.(248) and (287). Indeed,

$$\begin{aligned}
&\overset{\gamma_g}{\boldsymbol{R}}_2(a\wedge b)\cdot(c\wedge w) \\
&= \overset{\gamma_g}{\mathbf{R}}_3(a,b,c,w) \\
&= \overset{\varkappa\eta}{\mathbf{R}}_3(a,b,\boldsymbol{h}(c),\boldsymbol{h}(w)) \\
&= \overset{\varkappa\eta}{\boldsymbol{R}}_2(a\wedge b)\cdot(\boldsymbol{h}(c)\wedge\boldsymbol{h}(w)) \\
&= \underline{h}^{\dagger}\overset{\varkappa\eta}{\boldsymbol{R}}_2(a\wedge b)\cdot(c\wedge w),
\end{aligned}$$

which implies

$$\overset{\gamma_g}{\boldsymbol{R}}_2(a\wedge b) = \underline{\boldsymbol{h}}^{\dagger}\overset{\varkappa\eta}{\boldsymbol{R}}_2(a\wedge b),$$

i.e.,

$$\overset{\gamma_g}{\boldsymbol{R}}(B) = \underline{\boldsymbol{h}}^{\dagger}\overset{\varkappa\eta}{\boldsymbol{R}}_2(B). \qquad (288)$$

We get now $\overset{\varkappa\eta}{\rho}(a,b,c)$ and $\overset{\varkappa\eta}{\boldsymbol{R}}_2(a\wedge b)$ in terms of the $\varkappa\eta$-gauge rotation field $\overset{\varkappa\eta}{\Omega}$ of the Minkowski-Cartan structure; For $\overset{\varkappa\eta}{\rho}(a,b,c)$ we have (recallin Eq.(274)):

$$\begin{aligned}
\overset{\varkappa\eta}{\rho}(a,b,c) &= [\overset{\varkappa}{\mathcal{D}}_a,\overset{\varkappa}{\mathcal{D}}_b]c - \overset{\varkappa}{\mathcal{D}}_{[a,b]}c \\
&= (a\cdot\partial\overset{\varkappa\eta}{\Omega}(b) - b\cdot\partial\overset{\varkappa\eta}{\Omega}(a) - \overset{\varkappa\eta}{\Omega}([a,b]) + \overset{\varkappa\eta}{\Omega}(a)\underset{\eta}{\times}\overset{\varkappa\eta}{\Omega}(b))\underset{\eta}{\times}c \\
\overset{\varkappa\eta}{\rho}(a,b,c) &= ((a\cdot\partial\overset{\varkappa\eta}{\Omega})(b) - (b\cdot\partial\overset{\varkappa\eta}{\Omega})(a) + \overset{\varkappa\eta}{\Omega}(a)\underset{\eta}{\times}\overset{\varkappa\eta}{\Omega}(b))\underset{\eta}{\times}c. \qquad (289)
\end{aligned}$$

For $\overset{\varkappa\eta}{\boldsymbol{R}}_2(a\wedge b)$, utilizing the factorization theorem for $\mathbf{R}_3$, Eq.(248),the Eq.(289) and the multiform identities $B\underset{g}{\times}b = B\underset{g}{\llcorner}b$ and $(B\underset{g}{\llcorner}b)\underset{g}{\cdot}a = B\underset{g}{\cdot}(a\wedge b)$, with

$B \in \sec \bigwedge^2 T^*U$ and $a, b \in \sec \bigwedge^1 T^*U$, we have:

$$\begin{aligned}\overset{\varkappa\eta}{\boldsymbol{R}}_2(a \wedge b) \cdot (c \wedge w) &= \overset{\varkappa\eta}{\mathbf{R}}_3(a, b, c, w) \\ &= -\overset{\varkappa\eta}{\rho}(a, b, c) \underset{\eta}{\cdot} w \\ &= -((a \cdot \partial \overset{\varkappa\eta}{\Omega}(b) - b \cdot \partial \overset{\varkappa\eta}{\Omega}(a) - \overset{\varkappa\eta}{\Omega}([a, b]) + \overset{\varkappa\eta}{\Omega}(a) \underset{\eta}{\times} \overset{\varkappa\eta}{\Omega}(b)) \underset{\eta}{\times} c) \underset{\eta}{\cdot} w \\ &= (a \cdot \partial \overset{\varkappa\eta}{\Omega}(b) - b \cdot \partial \overset{\varkappa}{\Omega}(a) - \overset{\varkappa\eta}{\Omega}([a, b]) + \overset{\varkappa}{\Omega}(a) \underset{\eta}{\times} \overset{\varkappa\eta}{\Omega}(b)) \cdot \underline{\eta}(c \wedge w).\end{aligned}$$

Relabeling $c \to \eta(c)$ and $w \to \eta(w)$, and recalling that $\eta^2 = i_d$, we have

$$\overset{\varkappa\eta}{\boldsymbol{R}}_2(a \wedge b) \cdot \underline{\eta}(c \wedge w) = (a \cdot \partial \overset{\varkappa\eta}{\Omega}(b) - b \cdot \partial \overset{\varkappa\eta}{\Omega}(a) - \overset{\varkappa\eta}{\Omega}([a, b]) + \overset{\varkappa\eta}{\Omega}(a) \underset{\eta}{\times} \overset{\varkappa\eta}{\Omega}(b)) \cdot (c \wedge w),$$

which implies that

$$\underline{\eta}\overset{\varkappa\eta}{\boldsymbol{R}}_2(a \wedge b) = a \cdot \partial \overset{\varkappa\eta}{\Omega}(b) - b \cdot \partial \overset{\varkappa\eta}{\Omega}(a) - \overset{\varkappa\eta}{\Omega}([a, b]) + \overset{\varkappa\eta}{\Omega}(a) \underset{\eta}{\times} \overset{\varkappa}{\Omega}(b). \tag{290}$$

4.8.4 Gauge Extensor Fields Associated to a Lorentz-Cartan *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$

We introduce now three smooth *gauge* extensor fields on U_o obtained from the Minkowski-Cartan *MCGSS* $(U_o, \eta, \varkappa)$ and which encodes all the information encoded in the Lorentz-Cartan *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$ where $\boldsymbol{g} = \boldsymbol{h}^\dagger \eta \boldsymbol{h}$. First we define

(i) The $(2, 2)$-extensor field, $B \mapsto \overset{\gamma_h}{\mathcal{R}}_2(B)$, given by

$$\overset{\gamma_h}{\mathcal{R}}_2(B) = \underline{\eta}\overset{\varkappa\eta}{\boldsymbol{R}}_2(B), \tag{291}$$

which is called the *Riemann gauge field for* $(U_o, \boldsymbol{g}, \gamma)$, and the reason for that name will be clear in a while.

(ii) The $(1, 1)$-extensor field, $b \mapsto \overset{\gamma_h}{\mathcal{R}}_1(b)$, given by

$$\overset{\gamma_h}{\mathcal{R}}_1(b) = \boldsymbol{h}^{\clubsuit}(\partial_a) \lrcorner \overset{\gamma_h}{\mathcal{R}}_2(a \wedge b), \tag{292}$$

called the Ricci *gauge field for* $(U_o, \boldsymbol{g}, \gamma)$.

(iii) The scalar field $\overset{\gamma_h}{\mathcal{R}}$, given by by

$$\overset{\gamma_h}{\mathcal{R}} = \boldsymbol{h}^{\clubsuit}(\partial_b) \cdot \overset{\gamma_h}{\mathcal{R}}_1(b) = \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot \overset{\gamma_h}{\mathcal{R}}_2(a \wedge b), \tag{293}$$

called the *gauge Ricci scalar field for* $(U_o, \boldsymbol{g}, \gamma)$.

(iv) The $(1, 1)$-extensor field, $a \mapsto \overset{\gamma_h}{\mathcal{G}}(a)$, given by

$$\overset{\gamma_h}{\mathcal{G}}(a) = \overset{\gamma_h}{\mathcal{R}}_1(a) - \frac{1}{2}\boldsymbol{h}(a)\overset{\gamma_h}{\mathcal{R}}, \tag{294}$$

called *the Einstein gauge field for* $(U_o, \boldsymbol{g}, \gamma)$.

Proposition 4.5 The Riemann gauge field has the fundamental property

$$\overset{\gamma_h}{\mathcal{R}}_2(a \wedge b) = a \cdot \partial \overset{\varkappa\eta}{\Omega}(b) - b \cdot \partial \overset{\varkappa\eta}{\Omega}(a) - \overset{\varkappa\eta}{\Omega}([a, b]) + \overset{\varkappa\eta}{\Omega}(a) \underset{\eta}{\times} \overset{\varkappa\eta}{\Omega}(b) \tag{295}$$

Proof Eq.(295) is an immediate consequence of Eq.(291)and Eq.(290). ■

Proposition 4.6 The Ricci scalar field $\overset{\gamma_g}{R}$ (associated with $(U_o, \boldsymbol{g}, \gamma)$) is equal to $\overset{\gamma_h}{\mathcal{R}}$, i.e.,

$$\overset{\gamma_g}{R} = \overset{\gamma_h}{\mathcal{R}} \tag{296}$$

Proof By a simple algebraic manipulation of Eq.(249), utilizing the Theorem 4.1, the Eq.(288) and Eq.(289), we get that

$$\begin{aligned}
\overset{\gamma_g}{R} &= \underline{g}^{-1}(\partial_a \wedge \partial_b) \cdot \overset{\gamma_g}{\boldsymbol{R}}_2(a \wedge b) = \underline{\boldsymbol{h}}^{-1} \underline{\eta} \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot \underline{\boldsymbol{h}}^\dagger \overset{\varkappa\eta}{\boldsymbol{R}}_2(a \wedge b) \\
&= \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot \underline{\eta} \overset{\varkappa\eta}{\boldsymbol{R}}_2(a \wedge b) = \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot \overset{\gamma_h}{\mathcal{R}}_2(a \wedge b),
\end{aligned}$$

i.e., by Eq.(293), $\overset{\gamma_g}{R} = \overset{\gamma_h}{\mathcal{R}}$. ■

Proposition 4.7 The Ricci $(1,1)$-extensor field, $b \mapsto \overset{\gamma_g}{\boldsymbol{R}}_1(b)$, and Einstein $(1,1)$-extensor field, $a \mapsto \overset{\gamma_g}{\boldsymbol{G}}(a)$, associated to the Lorentz-Cartan *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$ are related to the gauge fields $b \mapsto \overset{\gamma_h}{\mathcal{R}}_1(b)$ and $a \mapsto \overset{\gamma_h}{\mathcal{G}}(a)$ by

$$\overset{\gamma_g}{\boldsymbol{R}}_1(b) = \boldsymbol{h}^\dagger \eta \overset{\gamma_h}{\mathcal{R}}_1(b), \tag{297}$$

$$\overset{\gamma_g}{\boldsymbol{G}}(a) = \boldsymbol{h}^\dagger \eta \overset{\gamma_h}{\mathcal{G}}(a). \tag{298}$$

Proof By simple algebraic manipulation of Eq.(249), utilizing Eq.(288), Eq. (291), Eq.(56) and the Theorem 4.1, we get that

$$\begin{aligned}
\overset{\gamma_g}{\boldsymbol{R}}_1(b) &= \boldsymbol{g}^{-1}(\partial_a) \lrcorner \overset{\gamma_g}{\boldsymbol{R}}_2(a \wedge b) = \boldsymbol{g}^{-1}(\partial_a) \lrcorner \underline{\boldsymbol{h}}^\dagger \overset{\varkappa\eta}{\boldsymbol{R}}_2(a \wedge b) \\
&= \boldsymbol{g}^{-1}(\partial_a) \lrcorner \underline{\boldsymbol{h}}^\dagger \underline{\eta} \overset{\gamma_h}{\mathcal{R}}_2(a \wedge b) = \boldsymbol{h}^\dagger \eta (\eta \boldsymbol{h} \boldsymbol{h}^{-1} \eta \boldsymbol{h}^{\clubsuit}(\partial_a) \lrcorner \overset{\gamma_h}{\mathcal{R}}_2(a \wedge b)) \\
&= \boldsymbol{h}^\dagger \eta (\boldsymbol{h}^{\clubsuit}(\partial_a) \lrcorner \overset{\gamma_h}{\mathcal{R}}_2(a \wedge b)),
\end{aligned}$$

and recalling Eq.(292) we have that $\overset{\gamma_g}{\boldsymbol{R}}_1(b) = \boldsymbol{h}^\dagger \eta \overset{\gamma_h}{\mathcal{R}}_1(b)$.

Also, putting Eqs.(297) and (296) in Eq.(252) and utilizing the Theorem4.1, we get

$$\begin{aligned}
\overset{\gamma_g}{\boldsymbol{G}}(a) &= \overset{\gamma_g}{\boldsymbol{R}}_1(a) - \frac{1}{2} \boldsymbol{g}(a) \overset{\gamma_g}{R} = \boldsymbol{h}^\dagger \eta \overset{\gamma_h}{\mathcal{R}}_1(a) - \frac{1}{2} \boldsymbol{g}(a) \overset{\gamma_h}{\mathcal{R}} \\
&= \boldsymbol{h}^\dagger \eta \overset{\gamma_h}{\mathcal{R}}_1(a) - \frac{1}{2} \boldsymbol{h}^\dagger \eta \boldsymbol{h}(a) \overset{\gamma_h}{\mathcal{R}} = \boldsymbol{h}^\dagger \eta (\overset{\gamma_h}{\mathcal{R}}(a) - \frac{1}{2} \boldsymbol{h}(a) \overset{\gamma_h}{\mathcal{R}}),
\end{aligned}$$

and recalling Eq.(294) it follows that $\overset{\gamma_g}{\boldsymbol{G}}(a) = \boldsymbol{h}^{\dagger}\eta\overset{\gamma_h}{\mathcal{G}}(a)$. ■

The last results may be interpreted by saying that the plastic distortion field $\boldsymbol{h}$ living on the Minkowski *MCGSS* $(U_o, \eta, \mathring{\varkappa})$ deforms its natural parallel transport rule, thus generating a Minkowski-Cartan *MCGSS* $(U_o, \eta, \varkappa)$[44]. Equivalently, we may say that the field $\boldsymbol{h}$ generates an effective Lorentzian metric (extensor) $\boldsymbol{g}$ and that the Lorentz *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$ is gauge equivalent to the Minkowski-Cartan *MCGSS* $(U_o, \eta, \varkappa)$. This is the case because the Riemann gauge field $\overset{\gamma_h}{\mathcal{R}}(B)$ *encodes all the information contained in the Ricci* $(1,1)$-extensor field $\overset{\gamma_g}{\boldsymbol{R}}_1(a)$, in the Ricci scalar field $\overset{\gamma_g}{R}$ and in the Einstein $(1,1)$-extensor field $\overset{\gamma_g}{\boldsymbol{G}}(a)$ of the Lorentz-Cartan *MCGSS.* Moreover, we have the following proposition.

Proposition 4.8 The Ricci scalar field $\overset{\gamma_h}{\mathcal{R}}$ $(= \overset{\gamma_g}{R})$ is a *scalar* $\boldsymbol{h}^{\clubsuit}$*-divergent* of the Riemann gauge field gauge de Riemann $\overset{\gamma_h}{\mathcal{R}}_2(B)$, i.e.,

$$
\begin{aligned}
\overset{\gamma_h}{\mathcal{R}} &= \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot \overset{\gamma_h}{\mathcal{R}}_2(a \wedge b) \\
&= \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot (a \cdot \partial \overset{\varkappa\eta}{\Omega}(b) - b \cdot \partial \overset{\varkappa\eta}{\Omega}(a) - \overset{\varkappa\eta}{\Omega}([a,b]) + \overset{\varkappa\eta}{\Omega}(a) \underset{\eta}{\times} \overset{\varkappa\eta}{\Omega}(b)) \\
\overset{\gamma_h}{\mathcal{R}} &= \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot ((a \cdot \partial \overset{\varkappa\eta}{\Omega})(b) - (b \cdot \partial \overset{\varkappa\eta}{\Omega})(a) + \overset{\varkappa\eta}{\Omega}(a) \underset{\eta}{\times} \overset{\varkappa\eta}{\Omega}(b)).
\end{aligned} \tag{299}
$$

The proof of Proposition 4.8 follows trivially from the previous results.

Remark 4.5 Proposition 4.8 shows that $\overset{\gamma_h}{\mathcal{R}}$ may be interpreted as a *scalar functional* of the plastic gauge distortion field $\boldsymbol{h}$ an the $\varkappa\eta$-gauge rotation field $\overset{\varkappa\eta}{\Omega}$ (and their directional derivatives $\cdot\partial \overset{\varkappa\eta}{\Omega}$). However to get a simple theory for the gravitational field we need an additional result which is valid for a Lorentz *MCGSS* $(U_o, \boldsymbol{g}, \lambda)$

4.8.5 Lorentz *MCGSS* as *h*-Deformation of a Particular Minkowski-Cartan *MCGSS*

Let $(M, \boldsymbol{g}, D, \tau_g, \uparrow)$ be a Lorentzian spacetime structure as defined in the beginning of Section 1. We already called the triple $(M, \boldsymbol{g}, D)$ Lorentz *MCGSS.* As in the previous sections, let $\mathcal{U} \subset M$, U be the canonical vector space and $U_o \subset U$ the representative of the points of $\mathcal{U}$. Moreover let $\boldsymbol{g}$ be the metric extensor on U_o (which represents $\boldsymbol{g}$) and λ the *connection* 2*-extensor* on U_o representing the de Levi-Civita connection D. Under those conditions we also say that $(U_o, \boldsymbol{g}, \lambda)$ is a Lorentz *MCGSS.*

Observe that the Lorentz $(U_o, \boldsymbol{g}, \lambda)$ *MCGSS* is a particular Lorentz-Cartan *MCGSS* $(U_o, \boldsymbol{g}, \gamma)$ where the connection 2-extensor field is $\gamma = \lambda$ (the Levi-Civita connection 2-extensor field).

[44]The way in which $\mathring{\mathcal{D}}_a$ and $\mathcal{D}_a$ are related is a particular case of the way that two different connections defined on a manifold M are related, see details, e.g., in [82]

Let D_a be the covariant derivative operator defined by $(U_o, \boldsymbol{g}, \lambda)$ and let $a, b \in \sec \bigwedge^1 T^*U$. Under those conditions we shall use the following *notation* (recall Eq.(276))

$$D_a b = a \cdot \partial b + \frac{1}{2}\boldsymbol{g}^{-1}(a \cdot \partial \boldsymbol{g})(b) + \omega(a) \times \boldsymbol{g}(b), \tag{300}$$

where the $\boldsymbol{g}$-gauge rotation field ω of the $(U_o, \boldsymbol{g}, \lambda)$ *GSS* is given by the formula [33]

$$\omega(a) = -\frac{1}{2}\underline{\boldsymbol{g}}^{-1}(\partial_b \wedge \partial_c)A(a, b, c), \tag{301}$$

where

$$A(a, b, c) \equiv \frac{1}{2}a\cdot(b\cdot\partial\boldsymbol{g}(c) - c\cdot\partial\boldsymbol{g}(b) - \boldsymbol{g}([b, c])) = \frac{1}{2}a\cdot((b\cdot\partial\boldsymbol{g})(c) - (c\cdot\partial\boldsymbol{g})(b)), \tag{302}$$

is a scalar 3-extensor field (obviously antisymmetric with respect to the second and third variables).

If $\boldsymbol{g} = \boldsymbol{h}^\dagger \eta \boldsymbol{h}$ then according to the Theorem 4.3 we may interpret the Lorentz *MCGSS* $(U_o, \boldsymbol{g}, \lambda)$ as a deformation of a *particular* Minkowski-Cartan *MCGSS* which will be denoted by (U_o, η, μ). Indeed, in this case we have the following result:

Proposition 4.9 [33] If the covariant derivative operator defined by (U_o, η, μ) is denoted $\mathcal{D}_a$ then

$$D_a b = \boldsymbol{h}^{-1}(\mathcal{D}_a \boldsymbol{h}(b)), \tag{303}$$

where

$$\mathcal{D}_a b := a \cdot \partial b + \ \Omega(a) \underset{\eta}{\times} b \tag{304}$$

and the η-gauge rotation operator Ω of the Minkowski-Cartan (U_o, η, μ) is given by

$$\Omega(a) = -\frac{1}{2}\underline{\eta}^{-1}(\partial_b \wedge \partial_c)[a, \boldsymbol{h}^{-1}(b), \boldsymbol{h}^{-1}(c)], \tag{305}$$

where $[x, y, z]$ which has been defined by Eq.(222) is the first Christoffel operator associated to the Lorentz metric extensor $\boldsymbol{g} = \boldsymbol{h}^\dagger \eta \boldsymbol{h}$.

Remark 4.6 It is absolutely clear form the above formulas that the biform field $\Omega(a)$ may be interpreted as a *biform functional* of the plastic distortion gauge field $\boldsymbol{h}$ of its first directional derivatives $\cdot\partial\boldsymbol{h}$ and of its second order directional derivatives $\cdot\partial\cdot\partial\boldsymbol{h}$. It is that result that summed to the one recalled in Remark 4.5 that permits us to formulate a gravitational theory on Minkowski spacetime where this field is the plastic distortion field $\boldsymbol{h}$. This is indeed the case because if R is the curvature scalar in Einstein's *GRT* (the same as R in the Lorentz *GSS* $(U_o, \boldsymbol{g}, \lambda)$), then according to Eq.(296) (where now $\gamma = \lambda$), $R \equiv \overset{\lambda_g}{R} = \overset{\lambda_h}{\mathcal{R}}$ is a *scalar functional* of $\boldsymbol{h}^\clubsuit$, $\cdot\partial\boldsymbol{h}^\clubsuit$ and $\cdot\partial\cdot\partial\boldsymbol{h}^\clubsuit$.

5 Gravitation as Plastic Distortion of the Lorentz Vacuum

The Remark 4.6 clearly reveals the way that the dynamics of the plastic gauge deformations $\boldsymbol{h}^{\clubsuit}$ of the Lorentz vacuum must be described in a world that unless experimental facts (and none exists until now, for the best of our knowledge) demonstrate the contrary must be described by an event manifold $M \simeq \mathbb{R}^4$. We elaborate this point as follows.

Let $(M \simeq \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_\eta, \uparrow)$ be the structure representing Minkowski spacetime as defined in Section 1. In our theory we suppose that all physical fields and/or particles live and interact in the arena defined by Minkowski spacetime. Now, let $(M \simeq \mathbb{R}^4, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}, \uparrow)$ be a Lorentzian manifold that as we know represent a particular gravitational field in Einstein's *GRT*. Let moreover η',$\mathsf{g} \in \sec T_0^2 M$ be the metric tensors on the cotangent bundle such that in, the global chart with global coordinates [45] $\{\mathrm{x}^\alpha\}$ associated to a chart $(M, \phi)_o$ where

$$\boldsymbol{\eta} = \eta_{\alpha\beta} d\mathrm{x}^\alpha \otimes d\mathrm{x}^\beta, \quad \boldsymbol{g} = g_{\alpha\beta} d\mathrm{x}^\alpha \otimes d\mathrm{x}^\beta \tag{306}$$

we have

$$\eta' = \eta^{\alpha\beta} \frac{\partial}{\partial \mathrm{x}^\alpha} \otimes \frac{\partial}{\partial \mathrm{x}^\beta}, \qquad \mathsf{g} = g^{\alpha\beta} \frac{\partial}{\partial \mathrm{x}^\alpha} \otimes \frac{\partial}{\partial \mathrm{x}^\beta} \tag{307}$$

with $\eta_{\alpha\beta}\eta^{\beta\nu} = \delta_\alpha^\nu$ and $g_{\alpha\beta}g^{\beta\nu} = \delta_\alpha^\nu$. Let also η be the extensor field corresponding to η, i.e.,

$$\eta'(d\mathrm{x}^\alpha, d\mathrm{x}^\beta) = \eta(d\mathrm{x}^\alpha) \cdot d\mathrm{x}^\beta \tag{308}$$

where in Eq.(308) the symbol $\cdot$ denotes the canonical scalar product on $M \simeq \mathbb{R}^4$, i.e., $d\mathrm{x}^\alpha \cdot d\mathrm{x}^\beta = \delta^{\alpha\beta}$.

Since $M \simeq \mathbb{R}^4$, utilizing the global coordinates $\{\mathrm{x}^\alpha\}$ we have immediately that $M \simeq U_o \simeq U$ (where U_o and U are as previously introduced) and the equivalence classes of the $d\mathrm{x}^\alpha$ are $\boldsymbol{\vartheta}^\alpha := [d\mathrm{x}^\alpha]$, which $\{\boldsymbol{\vartheta}^\alpha\}$ defining a basis[46] for U.

Its reciprocal basis will be denoted $\{\vartheta_\alpha\}$. In this case given a smooth invertible extensor field $\mathbf{h} : \sec \bigwedge^1 TM \to \sec \bigwedge^1 TM$ its representative on U is the smooth extensor field $\boldsymbol{h} : \sec \bigwedge^1 T^*U \to \sec \bigwedge^1 T^*U$. Recall moreover that the metric tensor $\boldsymbol{g}$ is represented on U by the extensor field $\boldsymbol{g} : \sec \bigwedge^1 T^*U \to \sec \bigwedge^1 T^*U$,

$$\boldsymbol{g} = \boldsymbol{h}^\dagger \eta \boldsymbol{h}, \tag{309}$$

whereas the representative of g is

$$\boldsymbol{g}^{-1} = \boldsymbol{h}^{-1} \eta \boldsymbol{h}^{-1\dagger} \tag{310}$$

[45] Called coordinates in the Einstein-Lorentz-Poincaré gauge.

[46] The canonical reciprocal basis of $\{\boldsymbol{\vartheta}^\alpha\}$ of U is denoted by $\{\vartheta_\alpha\}$. The basis $\{\boldsymbol{\vartheta}_\alpha\}$ of U is defined by the condition $\boldsymbol{\vartheta}^\alpha \underset{\eta^{-1}}{\cdot} \boldsymbol{\vartheta}_\beta = \boldsymbol{\vartheta}^\alpha \underset{\eta}{\cdot} \boldsymbol{\vartheta}_\beta = \delta_\beta^\alpha$. It may be called the η-reciprocal basis of $\{\boldsymbol{\vartheta}^\alpha\}$.

with (for $\mathbf{x} \in U$ and $x \in M$, $\mathbf{x} = \phi(x)$)

$$\boldsymbol{g}^{-1}\big|_{\mathbf{x}}(\boldsymbol{\vartheta}^{\alpha}) \cdot \boldsymbol{\vartheta}^{\beta} := \mathtt{g}(d\mathrm{x}^{\alpha}, d\mathrm{x}^{\beta})\big|_{x}, \tag{311}$$

In what follows we present a gravitational theory where that field which *lives* in Minkowski spacetime is represented by smooth invertible extensor field $\boldsymbol{h}$: $\sec \bigwedge^1 T^*U \to \sec \bigwedge^1 T^*U$ describing (where it is *not* the identity) a plastic deformation of the Lorentz vacuum, putting it in a 'excited' state which may be described by Minkowski-Cartan *MCGSS* $(M, \eta, \varkappa)$ which as we already know is gauge equivalent to the Lorentz *MCGSS* $(M, \boldsymbol{g}, \gamma)$. We will give the Lagrangian encoding the dynamics of $\boldsymbol{h}$ and its interaction with the matter fields and will derive its equation of motion.

5.1 Lagrangian for the Free $\boldsymbol{h}^{\clubsuit}$ Field

Under the conditions established above we postulate that the dynamics of the gravitational field $\boldsymbol{h}$: $\sec \bigwedge^1 T^*U \to \sec \bigwedge^1 T^*U$ is encoded in the following Lagrangian, which using notations introduced in the previous section, is written as

$$[\boldsymbol{h}^{\clubsuit}, \cdot\partial\boldsymbol{h}^{\clubsuit}, \cdot\partial\cdot\partial\boldsymbol{h}^{\clubsuit}] \mapsto \mathfrak{L}_{EH}[\boldsymbol{h}^{\clubsuit}, \cdot\partial\boldsymbol{h}^{\clubsuit}, \cdot\partial\cdot\partial\boldsymbol{h}^{\clubsuit}] := \frac{1}{2}\mathcal{R}\det[\boldsymbol{h}], \tag{312}$$

where recalling Eq.(299) and Eq.(305)) we see that $\mathcal{R} \equiv \overset{\lambda_h}{\mathcal{R}}$ may indeed be expressed as a scalar functional field of extensor variables $\boldsymbol{h}^{\clubsuit}$, $\cdot\partial\boldsymbol{h}^{\clubsuit}$ and $\cdot\partial\cdot\partial\boldsymbol{h}^{\clubsuit}$, i.e.,

$$\begin{aligned}\mathcal{R} &= \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot \mathcal{R}_2(a \wedge b) \\ &= \underline{\boldsymbol{h}}^*(\partial_a \wedge \partial_b) \cdot (a \cdot \partial(\Omega b) - b \cdot \partial\Omega(a) - \Omega([a,b]) + \Omega(a) \underset{\eta}{\times} \Omega(b)),\end{aligned} \tag{313}$$

where the biform field $\Omega(a)$ given by Eq.(301) is also expressed as a functional for the extensor fields $\boldsymbol{h}^{\clubsuit}$, $\cdot\partial\boldsymbol{h}^{\clubsuit}$ and $\cdot\partial\cdot\partial\boldsymbol{h}^{\clubsuit}$.

To continue we must express also $\det[\boldsymbol{h}]$ as a scalar functional of $\boldsymbol{h}^{\clubsuit}$, i.e.,

$$\det[\boldsymbol{h}] = \frac{1}{\det[\boldsymbol{h}^{\clubsuit}]}. \tag{314}$$

The action functional for the field $\boldsymbol{h}^{\clubsuit}$ on U is

$$\begin{aligned}\mathcal{A} &= \int_U \mathfrak{L}_{eh}[\boldsymbol{h}^{\clubsuit}, \cdot\partial\boldsymbol{h}^{\clubsuit}, \cdot\partial\cdot\partial\boldsymbol{h}^{\clubsuit}]\ \tau \\ &= \frac{1}{2}\int_U \mathcal{R}\det[\boldsymbol{h}]\ \tau,\end{aligned} \tag{315}$$

where τ is an arbitrary Euclidean volume element, e.g., we can take $\tau = d\mathrm{x}^0 \wedge d\mathrm{x}^1 \wedge d\mathrm{x}^2 \wedge d\mathrm{x}^3$

Let $\boldsymbol{\delta}_{h^{\clubsuit}}^{w}$ be the variational operator with respect to $\boldsymbol{h}^{\clubsuit}$ in the direction of the smooth $(1,1)$-extensor field w, such that $w|_{\partial U} = 0$ and $\cdot\partial w|_{\partial U} = 0$.

5.1.1 Equations of Motion for $\boldsymbol{h}^{\clubsuit}$

As usual in the Lagrangian formalism we suppose that the dynamics of $\boldsymbol{h}^{\clubsuit}$ is given by the principle of stationary action,

$$\boldsymbol{\delta}^w_{\boldsymbol{h}^{\clubsuit}} \int_U \mathcal{L}_{eh}[\boldsymbol{h}^{\clubsuit}, \cdot\partial \boldsymbol{h}^{\clubsuit}, \cdot\partial \cdot \partial \boldsymbol{h}^{\clubsuit}]\tau = \int_U \boldsymbol{\delta}^w_{\boldsymbol{h}^{\clubsuit}} (\mathcal{R} \det[\boldsymbol{h}])\ \tau = 0, \tag{316}$$

which implies in a functional differential equation (Euler-Lagrange equation) for $\boldsymbol{h}^{\clubsuit}$ that we now derive.

As it is well known the solution of that problem implies in the use of variational formulas, the Gauss-Stokes theorem (for star shape regions), i.e.,

$$\int_U \partial \cdot a\ \tau = \int_{\partial U} \boldsymbol{\vartheta}^{\mu} \cdot a\ \tau_{\mu}, \tag{317}$$

where τ_μ are 3-form fields on the boundary ∂U of U, a is a smooth 1-form field and also a fundamental lemma of integration theory, which says that if $\int_U A \cdot X$ $\tau = 0$ for all multiform field A then the $X = 0$.

Before starting the calculations we recall from Section 3.8 that any variational operator $\boldsymbol{\delta}^w_t$ satisfies the following rules. Let Φ, Ψ be arbitrary multiform functionals of the extensor field t. Moreover, let $\digamma$ be a scalar functional of the extensor field t with a a smooth 1-form field and φ an arbitrary function. Then:

$$\boldsymbol{\delta}^w_t (\Phi[t] * \Psi[t]) = (\boldsymbol{\delta}^w_t \Phi[t]) * \Psi[t] + \Phi[t] * (\boldsymbol{\delta}^w_t \Psi[t]), \tag{318a}$$

$$\boldsymbol{\delta}^w_t t(a) = w(a), \tag{318b}$$

$$\boldsymbol{\delta}^w_t (a \cdot \partial \Phi[t]) = a \cdot \partial(\boldsymbol{\delta}^w_t \Phi[t]), \tag{318c}$$

$$\boldsymbol{\delta}^w_t \varphi(\digamma[t]) = \varphi'(\digamma[t])(\boldsymbol{\delta}^w_t \digamma[t]), \tag{318d}$$

where in Eq.(318a) $*$ denotes here (as previously agreed) any one of the multiform products.

Utilizing Eq.(318a) in Eq.(312), and taking into account Eq.(313), we get

$$\begin{aligned} \boldsymbol{\delta}^w_{\boldsymbol{h}^{\clubsuit}} (\underline{\boldsymbol{h}}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot \mathcal{R}_2(a \wedge b) \det[\boldsymbol{h}]) &= (\boldsymbol{\delta}^w_{\boldsymbol{h}^{\clubsuit}} \underline{\boldsymbol{h}}^{\clubsuit}(\partial_a \wedge \partial_b)) \cdot \mathcal{R}_2(a \wedge b) \det[\boldsymbol{h}] \\ &+ \underline{\boldsymbol{h}}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot (\boldsymbol{\delta}^w_{\boldsymbol{h}^{\clubsuit}} \mathcal{R}_2(a \wedge b)) \det[\boldsymbol{h}] \\ &+ \underline{\boldsymbol{h}}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot \mathcal{R}_2(a \wedge b)(\boldsymbol{\delta}^w_{\boldsymbol{h}^{\clubsuit}} \det[\boldsymbol{h}]). \end{aligned} \tag{319}$$

We next obtain a functional identity for the scalar product in the first term

of Eq.(319). We have,

$$
\begin{aligned}
&(\boldsymbol{\delta}^w_{h^\clubsuit}\underline{\boldsymbol{h}}^\clubsuit(\partial_a\wedge\partial_b))\cdot\mathcal{R}_2(a\wedge b)\\
&=(\boldsymbol{\delta}^w_{\boldsymbol{h}^\clubsuit}\boldsymbol{h}^\clubsuit(\partial_a)\wedge\boldsymbol{h}^\clubsuit(\partial_b)+\boldsymbol{h}^\clubsuit(\partial_a)\wedge\boldsymbol{\delta}^w_{h^\clubsuit}\boldsymbol{h}^\clubsuit(\partial_b))\cdot\mathcal{R}_2(a\wedge b)\\
&=(w(\partial_a)\wedge\boldsymbol{h}^\clubsuit(\partial_b)+\boldsymbol{h}^\clubsuit(\partial_a)\wedge w(\partial_b))\cdot\mathcal{R}_2(a\wedge b)\\
&=(w(\partial_a)\wedge\boldsymbol{h}^\clubsuit(\partial_b))\cdot\mathcal{R}_2(a\wedge b)+(\boldsymbol{h}^\clubsuit(\partial_a)\wedge w(\partial_b))\cdot\mathcal{R}_2(a\wedge b)\\
&=(\boldsymbol{h}^\clubsuit(\partial_b)\wedge w(\partial_a))\cdot\mathcal{R}_2(b\wedge a)+(\boldsymbol{h}^\clubsuit(\partial_b)\wedge w(\partial_a))\cdot\mathcal{R}_2(b\wedge a)\\
&=2(\boldsymbol{h}^\clubsuit(\partial_b)\wedge w(\partial_a))\cdot\mathcal{R}_2(b\wedge a)\\
&=2w(\partial_a)\cdot(\boldsymbol{h}^\clubsuit(\partial_b)\lrcorner\mathcal{R}_2(b\wedge a)),
\end{aligned}
$$

i.e.,

$$
(\boldsymbol{\delta}^w_{h^\clubsuit}\underline{\boldsymbol{h}}^\clubsuit(\partial_a\wedge\partial_b))\cdot\mathcal{R}_2(a\wedge b)=2w(\partial_a)\cdot\mathcal{R}_1(a), \tag{320}
$$

where we utilized Eq.(318a) in the form $\boldsymbol{\delta}^w_t(\boldsymbol{\Phi}[t]\wedge\boldsymbol{\Psi}[t])=(\boldsymbol{\delta}^w_t\boldsymbol{\Phi}[t])\wedge\boldsymbol{\Psi}[t]+\boldsymbol{\Phi}[t])\wedge(\boldsymbol{\delta}^w_t\ominus[t])$, Eq.(318b) and Eq.(292) which defines the Ricci gauge field.

We next obtain a second functional identity for the scalar product in the second term[47] of Eq.(319),

$$
\begin{aligned}
\underline{\boldsymbol{h}}^\clubsuit(\partial_a\wedge\partial_b)\cdot\boldsymbol{\delta}^w_{h^\clubsuit}\mathcal{R}_2(a\wedge b)&=\underline{\boldsymbol{h}}^\clubsuit(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)\cdot\boldsymbol{\delta}^w_{h^\clubsuit}\mathcal{R}_2(\vartheta_\mu\wedge\vartheta_\nu)\\
&=\underline{\boldsymbol{h}}^\clubsuit(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)\cdot\boldsymbol{\delta}^w_{h^\clubsuit}(\vartheta_\mu\cdot\partial\Omega(\vartheta_\nu)-\vartheta_\nu\cdot\partial\Omega(\vartheta_\mu)\\
&+\Omega(\vartheta_\mu)\underset{\eta}{\times}\Omega(\vartheta_\nu))\\
&=\underline{\boldsymbol{h}}^\clubsuit(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)\cdot(\vartheta_\mu\cdot\partial\boldsymbol{\delta}^w_{h^\clubsuit}\Omega(\vartheta_\nu)-\vartheta_\nu\cdot\partial\boldsymbol{\delta}^w_{h^\clubsuit}\Omega(\vartheta_\mu)\\
&+\delta^w_{\boldsymbol{h}^\clubsuit}\Omega(\vartheta_\mu)\underset{\eta}{\times}\Omega(\vartheta_\nu)+\Omega(\vartheta_\mu)\underset{\eta}{\times}\boldsymbol{\delta}^w_{\boldsymbol{h}^\clubsuit}\Omega(\vartheta_\nu))\\
&=\underline{\boldsymbol{h}}^\clubsuit(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)\cdot(\mathcal{D}_{\vartheta_\mu}\boldsymbol{\delta}^w_{\boldsymbol{h}^\clubsuit}\Omega(\vartheta_\nu)-\mathcal{D}_{\vartheta_\nu}\boldsymbol{\delta}^w_{\boldsymbol{h}^\clubsuit}\Omega(\vartheta_\mu))\\
&=2\underline{\boldsymbol{h}}^\clubsuit(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)\cdot\mathcal{D}_{\vartheta_\mu}\delta^w_{h^\clubsuit}\Omega(\vartheta_\nu)\\
&=2\boldsymbol{h}^\clubsuit(\boldsymbol{\vartheta}^\nu)\cdot(\boldsymbol{h}^\clubsuit(\boldsymbol{\vartheta}^\mu)\lrcorner\mathcal{D}_{\vartheta_\mu}\delta^w_{h^\clubsuit}\Omega(\vartheta_\nu))\\
\underline{\boldsymbol{h}}^\clubsuit(\partial_a\wedge\partial_b)\cdot\boldsymbol{\delta}^w_{h^\clubsuit}\mathcal{R}_2(a\wedge b)&=2\boldsymbol{h}^\clubsuit(\vartheta^\nu)\cdot\boldsymbol{D}\lrcorner\boldsymbol{\delta}^w_{h^\clubsuit}\Omega(\vartheta_\nu).
\end{aligned} \tag{321}
$$

On those equations we utilized essentially Eq.(318c) once more the Eq.(318a), this time in the form, $\boldsymbol{\delta}^w_t(\boldsymbol{\Phi}[t]\underset{\eta}{\times}\boldsymbol{\Psi}[t])=(\boldsymbol{\delta}^w_t\boldsymbol{\Phi}[t])\underset{\eta}{\times}\boldsymbol{\Psi}[t]+\boldsymbol{\Phi}[t])\underset{\eta}{\times}(\boldsymbol{\delta}^w_t\boldsymbol{\Psi}[t])$, and the definition of the gauge divergent $\boldsymbol{D}\lrcorner X=\boldsymbol{h}^\clubsuit(\partial_a)\lrcorner\mathcal{D}_aX$, where $\mathcal{D}_a$ is the covariant derivative operator, which applied on a smooth multiform field X gives

$$
\mathcal{D}_aX:=a\cdot\partial X+\Omega(a)\underset{\eta}{\times}X. \tag{322}
$$

We may also write the right hand side of Eq.(317) which is a scalar product as the scalar divergent of a smooth 1-form field, utilizing the following identity (which is easily obtained if we take into account Eq.(266),

$$
\boldsymbol{D}\lrcorner X=\frac{1}{\det[\boldsymbol{h}]}\underline{\boldsymbol{h}}(\partial\lrcorner\det[\boldsymbol{h}]\underline{\boldsymbol{h}}^{-1}(X)). \tag{323}
$$

[47]Keep in mind that that $\{\vartheta_\mu\}$ is the canonical reciprocal basis of $\{\boldsymbol{\vartheta}^\mu\}$, i.e., $\boldsymbol{\vartheta}^\mu\cdot\vartheta_\nu=\delta^\mu_\nu$.

We then have

$$\begin{aligned}
\boldsymbol{h}^{\clubsuit}(\boldsymbol{\vartheta}^{\nu}) \cdot \boldsymbol{D}\lrcorner\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu}) &= \boldsymbol{h}^{\clubsuit}(\boldsymbol{\vartheta}^{\nu}) \cdot \frac{1}{\det[\boldsymbol{h}]}\boldsymbol{h}(\partial\lrcorner\det[\boldsymbol{h}]\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu}))) \\
&= \frac{1}{\det[\boldsymbol{h}]}\boldsymbol{\vartheta}^{\nu} \cdot \partial\lrcorner(\det[\boldsymbol{h}]\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu})) \\
&= \frac{1}{\det[\boldsymbol{h}]}\boldsymbol{\vartheta}^{\nu} \cdot (\boldsymbol{\vartheta}^{\mu}\lrcorner\vartheta_{\mu} \cdot \partial(\det[\boldsymbol{h}]\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu})))
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\det[\boldsymbol{h}]}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot \vartheta_{\mu} \cdot \partial(\det[\boldsymbol{h}]\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu})) \\
&= -\frac{1}{\det[\boldsymbol{h}]}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot \vartheta_{\mu} \cdot \partial(\det[\boldsymbol{h}]\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu})) \\
&= -\frac{1}{\det[\boldsymbol{h}]}\boldsymbol{\vartheta}^{\mu} \cdot (\boldsymbol{\vartheta}^{\nu}\lrcorner\vartheta_{\mu} \cdot \partial(\det[\boldsymbol{h}]\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu}))) \\
&= -\frac{1}{\det[\boldsymbol{h}]}\boldsymbol{\vartheta}^{\mu} \cdot \vartheta_{\mu} \cdot \partial(\boldsymbol{\vartheta}^{\nu}\lrcorner\det[\boldsymbol{h}]\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu}))) \\
&= -\frac{1}{\det[\boldsymbol{h}]}\partial \cdot (\det[\boldsymbol{h}]\boldsymbol{\vartheta}^{\nu}\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu}))).
\end{aligned}$$

i.e.,

$$\boldsymbol{h}^{\clubsuit}(\boldsymbol{\vartheta}^{\nu}) \cdot \boldsymbol{D}\lrcorner\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(\vartheta_{\nu}) = -\frac{1}{\det[\boldsymbol{h}]}\partial \cdot (\det[\boldsymbol{h}]\partial_{a}\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(a))). \tag{324}$$

Now, putting Eq.(324) in Eq.(321) we get an analogous of the well known *Palatini identity,* i.e.,

$$\underline{\boldsymbol{h}}^{\clubsuit}(\partial_{a} \wedge \partial_{b}) \cdot \boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\mathcal{R}_{2}(a \wedge b) = -2\frac{1}{\det[\boldsymbol{h}]}\partial \cdot (\det[\boldsymbol{h}]\partial_{a}\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\Omega(a))). \tag{325}$$

Next, we calculate the variation of $\det[\boldsymbol{h}]$ with respect to $\boldsymbol{h}^{\clubsuit}$ in the direction of w. In order to do so we utilize the chain rule given by Eq.(318d) and the variational formula (recall Eq.(180))

$$\boldsymbol{\delta}^{w}_{t}\det[t] = w(\partial_{a}) \cdot t^{\clubsuit}(a)\det[t], \tag{326}$$

which permit us to write

$$\begin{aligned}
\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\det[\boldsymbol{h}] &= \boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\frac{1}{\det[\boldsymbol{h}^{\clubsuit}]} \\
&= -\frac{1}{(\det[\boldsymbol{h}^{\clubsuit}])^{2}}\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\det[\boldsymbol{h}^{\clubsuit}] \\
&= -\frac{1}{(\det[\boldsymbol{h}^{\clubsuit}])^{2}}w(\partial_{a}) \cdot \boldsymbol{h}(a)\det[\boldsymbol{h}^{\clubsuit}] \\
&= -\frac{1}{\det[\boldsymbol{h}^{\clubsuit}]}w(\partial_{a}) \cdot \boldsymbol{h}(a) \\
\boldsymbol{\delta}^{w}_{\boldsymbol{h}^{\clubsuit}}\det[\boldsymbol{h}] &= -w(\partial_{a}) \cdot \boldsymbol{h}(a)\det[\boldsymbol{h}].
\end{aligned} \tag{327}$$

Finally, utilizing Eqs.(320), (325) and (327) in Eq.(319) we get the variation with respect to $\boldsymbol{h}^{\clubsuit}$ in the direction of w of the Lagrangian given by Eq.(312),

$$
\begin{aligned}
\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}(\mathcal{R}\det[\boldsymbol{h}]) &= 2w(\partial_a)\cdot\mathcal{R}_1(a)\det[\boldsymbol{h}] \\
&\quad - 2\frac{1}{\det[\boldsymbol{h}]}\partial\cdot(\det[\boldsymbol{h}]\partial_a\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}\Omega(a)))\det[\boldsymbol{h}] \\
&\quad - \underline{\boldsymbol{h}}^{\clubsuit}(\partial_a\wedge\partial_b)\cdot\mathcal{R}_2(a\wedge b)(w(\partial_a)\cdot\boldsymbol{h}(a)\det[\boldsymbol{h}]) \\
&= 2w(\partial_a)\cdot(\mathcal{R}_1(a)-\frac{1}{2}\boldsymbol{h}(a)\mathcal{R})\det[\boldsymbol{h}] - 2\partial\cdot(\det[\boldsymbol{h}]\partial_a\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}\Omega(a))) \\
\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}(\mathcal{R}\det[\boldsymbol{h}]) &= 2w(\partial_a)\cdot\mathcal{G}(a)\det[\boldsymbol{h}] - 2\partial\cdot(\det[\boldsymbol{h}]\partial_a\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}\Omega(a))). \qquad (328)
\end{aligned}
$$

In the last step we recalled the definition (Eq.(294)) of the Einstein gauge field $\mathcal{G}$ for the particular case of the Lorentz *MCGSS*.

Tanking into account Eq.(328), the contour problem for the dynamic variable $\boldsymbol{h}^{\clubsuit}$ becomes

$$
\int_U w(\partial_a)\cdot\mathcal{G}(a)\det[\boldsymbol{h}]\;\tau - \int_U \partial\cdot(\det[\boldsymbol{h}]\partial_a\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}\Omega(a)))\;\tau = 0, \tag{329}
$$

for all w satisfying the boundary conditions $w|_{\partial U}=0$ and $a\cdot\partial w|_{\partial U}=0$.

Utilizing then the Gauss-Stokes theorem with the above boundary conditions the second term of Eq.(329) may be integrated and gives

$$
\int_U \partial\cdot(\det[\boldsymbol{h}]\partial_a\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}\Omega(a)))\;\tau = \int_{\partial U}\det[\boldsymbol{h}]\boldsymbol{\vartheta}^{\mu}\cdot(\partial_a\lrcorner\underline{\boldsymbol{h}}^{-1}(\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}\Omega(a)))\;\tau_\mu = 0, \tag{330}
$$

since $\boldsymbol{\delta}_{\boldsymbol{h}^{\clubsuit}}^{w}\Omega(a)=0$ under the boundary conditions $w|_{\partial U}=0$ and $a\cdot\partial w|_{\partial U}=0$.

Thus, utilizing Eq.(330) in Eq.(329) it follows that

$$
\int_U w(\partial_a)\cdot\mathcal{G}(a)\det[\boldsymbol{h}]\;\tau = 0, \tag{331}
$$

for all w, and since w is arbitrary a fundamental lemma of integration theory yields

$$
\mathcal{G}(a)\det[\boldsymbol{h}] = 0, \tag{332}
$$

i.e.,

$$
\mathcal{G}(a) = \mathcal{R}_1(a) - \frac{1}{2}\boldsymbol{h}(a)\mathcal{R} = 0, \tag{333}
$$

which is the field equation for the distortion gauge field $\boldsymbol{h}$.

If we recall the relation between the Einstein $(1,1)$-extensor field, $a\mapsto\boldsymbol{G}(a)$ and the Einstein gauge field, $a\mapsto\mathcal{G}(a)$, (i.e., $\boldsymbol{G}(a)=\boldsymbol{h}^{\dagger}\eta\mathcal{G}(a)$), we get

$$
\boldsymbol{G}(a) = 0, \tag{334}
$$

which we recognize as equivalent to the free Einstein equation for $\boldsymbol{g}$.

5.1.2 Lagrangian for the Gravitational Field Plus Matter Field Including a Cosmological Constant Term

Since cosmological data seems to suggest an expansion of the universe (when it is described in terms of the Lorentzian spacetime model) we postulate here that the total Lagrangian describing the interaction of the gravitational field with matter is[48]

$$\mathfrak{L} = \mathfrak{L}_{eh} + \lambda \det[\boldsymbol{h}] + \mathfrak{L}_m \det[\boldsymbol{h}], \tag{335}$$

where λ is the cosmological constant and where $\mathfrak{L}_m$ is the matter Lagrangian. The equations of motion for $\boldsymbol{h}$ are obtained from the variational principle,

$$\int_U \boldsymbol{\delta}^w_{\boldsymbol{h}^\clubsuit}(\frac{1}{2}\mathcal{R} + \lambda + \mathfrak{L}_m) \det[\boldsymbol{h}]\ \tau = 0. \tag{336}$$

We then get defining conveniently the energy-momentum extensor of matter $T(a)$ by

$$(w(\partial_a) \cdot \eta \boldsymbol{h}^\clubsuit T(a)) \det[\boldsymbol{h}] := \boldsymbol{\delta}^w_{\boldsymbol{h}^\clubsuit}(\mathfrak{L}_m \det[\boldsymbol{h}]), \tag{337}$$

that

$$\mathcal{R}_1(a) - \frac{1}{2}\boldsymbol{h}(a)\mathcal{R} - \lambda \boldsymbol{h}(a) = -\eta \boldsymbol{h}^\clubsuit T(a). \tag{338}$$

Multiplying Eq.(338) on both sides by $\boldsymbol{h}^\dagger \eta$ we get:

$$\boldsymbol{G}(a) - \lambda \boldsymbol{g}(a) = -T(a). \tag{339}$$

6 Formulation of the Gravitational Theory in Terms of the Potentials $\mathfrak{g}^\alpha = \boldsymbol{h}^\dagger(\boldsymbol{\vartheta}^\alpha)$

Let $\{\mathrm{x}^\mu\}$ be global coordinates in the Einstein-Lorentz Poincaré gauge for $M \simeq \mathbb{R}^4$ and write as above

$$\boldsymbol{\vartheta}^\mu = d\mathrm{x}^\mu. \tag{340}$$

Then

$$\boldsymbol{\eta} = \eta_{\alpha\beta} \boldsymbol{\vartheta}^\alpha \otimes \boldsymbol{\vartheta}^\beta. \tag{341}$$

Define next the *gravitational potentials*[49]

$$\mathfrak{g}^\mu = \boldsymbol{h}^\dagger(\boldsymbol{\vartheta}^\mu) \in \sec \bigwedge^1 T^*U. \tag{342}$$

We immediately have that

$$\mathfrak{g}^\alpha \underset{g^{-1}}{\cdot} \mathfrak{g}^\beta = \boldsymbol{h}^{-1}\eta \boldsymbol{h}^\clubsuit \mathfrak{g}^\alpha \cdot \mathfrak{g}^\beta = \eta \boldsymbol{h}^\clubsuit \mathfrak{g}^\alpha \cdot \boldsymbol{h}^\clubsuit \mathfrak{g}^\beta \tag{343}$$

$$= \boldsymbol{h}^\clubsuit \mathfrak{g}^\alpha \underset{\eta^{-1}}{\cdot} \boldsymbol{h}^\clubsuit \mathfrak{g}^\beta =$$

$$= \boldsymbol{\vartheta}^\alpha \underset{\eta^{-1}}{\cdot} \boldsymbol{\vartheta}^\beta = \boldsymbol{\vartheta}^\alpha \underset{\eta}{\cdot} \boldsymbol{\vartheta}^\beta = \eta^{\alpha\beta}, \tag{344}$$

[48] Note that we are using geometrical units. The matter Lagrangian density is normaly written as $\mathfrak{L}_m = -\kappa \mathfrak{L}'_m$, where $\kappa = -8\pi G$, where G is Newton's gravitational constant.

[49] Take notice that in the calculations of this section we used the $\boldsymbol{g}$-reciprocal basis $\{\mathfrak{g}_\alpha\}$ of the basis $\{\mathfrak{g}^\mu\}$, i.e., $\mathfrak{g}^\mu \underset{g^{-1}}{\cdot} \mathfrak{g}_\alpha = \delta^\mu_\alpha$, i.e., $\mathfrak{g}_\alpha = \boldsymbol{h}^\dagger \eta(\vartheta_\alpha) = \eta_{\alpha\beta}\mathfrak{g}^\beta$.

i.e., the $\mathfrak{g}^{\mu}$ are $\boldsymbol{g}^{-1}$-orthonormal, which means that $\{\mathfrak{g}^{\mu}\}$ is a section of the $\boldsymbol{g}^{-1}$-orthonormal coframe bundle of $(M \simeq \mathbb{R}^4, \boldsymbol{g})$. Recalling now Eq.(296) which says that $R \equiv \overset{g}{R} = \mathcal{R}$ we can write the gravitational Lagrangian (Eq.(312)) as

$$\mathfrak{L}_{eh} := \frac{1}{2} R \det[\boldsymbol{h}], \tag{345}$$

and the Lagrangian density $\mathcal{L}_{eh} = \mathfrak{L}_{eh}\tau$ is then

$$\begin{aligned}\mathcal{L}_{eh} &= \frac{1}{2}\{\boldsymbol{h}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot \mathcal{R}_2(a \wedge b)\} \det \boldsymbol{h}\tau \\ &= \frac{1}{2}\{\boldsymbol{h}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot \eta \overset{\varkappa\eta}{\boldsymbol{R}}_2(a \wedge b)\}\tau_{g},\end{aligned} \tag{346}$$

where we have used that $\tau_{g} = \det \boldsymbol{h}\tau$ and Eq.(291) which says that $\mathcal{R}_2(a \wedge b) = \eta \overset{\mu\eta}{\boldsymbol{R}}_2(a \wedge b)$. Now, we write the 1-form fields a,b and the multiform derivatives ∂_a, ∂_b as

$$\begin{aligned} a &= a^{\kappa}\mathfrak{g}_{\kappa},\ b = b^{\kappa}\mathfrak{g}_{\kappa}, \\ \partial_a &= \mathfrak{g}^{\kappa}\frac{\partial}{\partial a^{\kappa}},\ \partial_b = \mathfrak{g}^{\kappa}\frac{\partial}{\partial b^{\kappa}}. \end{aligned} \tag{347}$$

Then, using Eq.(347) in Eq.(346) and taking into account Eq.(290) which says that $\overset{g}{\boldsymbol{R}}(B) \equiv \overset{\lambda g}{\boldsymbol{R}}(B) = \underline{\boldsymbol{h}}^{\dagger}\overset{\varkappa\eta}{\boldsymbol{R}}_2(B)$ we get

$$\begin{aligned} 2\mathcal{L}_{eh} &= \{\boldsymbol{h}^{\clubsuit}(\mathfrak{g}^{\kappa} \wedge \mathfrak{g}^{\iota}) \cdot \eta \overset{\mu\eta}{\boldsymbol{R}}_2(\mathfrak{g}_{\kappa} \wedge \mathfrak{g}_{\iota})\}\tau_{g} \\ &= \{\boldsymbol{h}^{\clubsuit}(\mathfrak{g}^{\kappa} \wedge \mathfrak{g}^{\iota}) \cdot \eta \boldsymbol{h}^{\clubsuit}\boldsymbol{h}^{\dagger}\overset{\mu\eta}{\boldsymbol{R}}_2(\mathfrak{g}_{\kappa} \wedge \mathfrak{g}_{\iota})\}\tau_{g} \\ &= \{\boldsymbol{h}^{-1}\eta\boldsymbol{h}^{\clubsuit}(\mathfrak{g}^{\kappa} \wedge \mathfrak{g}^{\iota}) \cdot \boldsymbol{h}_2^{\dagger}\overset{\mu\eta}{\boldsymbol{R}}_2(\mathfrak{g}_{\kappa} \wedge \mathfrak{g}_{\iota\mathbf{l}})\} \\ &= \{\boldsymbol{h}^{-1}\eta\boldsymbol{h}^{\clubsuit}(\mathfrak{g}^{\kappa} \wedge \mathfrak{g}^{\iota}) \cdot \overset{g}{\boldsymbol{R}}_2(\mathfrak{g}_{\kappa} \wedge \mathfrak{g}_{\iota})\}\tau_{g} \\ &\{(\mathfrak{g}^{\kappa} \wedge \mathfrak{g}^{\iota}) \underset{g^{-1}}{\cdot} \overset{g}{\boldsymbol{R}}_2(\mathfrak{g}_{\kappa} \wedge \mathfrak{g}_{\iota})\}\tau_{g} \end{aligned} \tag{348}$$

We now recall that from Eq.(248) it is

$$\overset{g}{\boldsymbol{R}}_2(\mathfrak{g}_{\kappa} \wedge \mathfrak{g}_{\iota}) \equiv \mathcal{R}_{\kappa\iota} \tag{349}$$

where $\mathcal{R}_{\kappa\iota} : U \to \bigwedge^2 U$ are the representatives of the *Cartan curvature 2-form fields.* Then recalling the definition of the Hodge star operator (Eq.(105)) we can write

$$\begin{aligned}\mathcal{L}_{eh} &= \frac{1}{2}\{(\mathfrak{g}^{\kappa} \wedge \mathfrak{g}^{\iota}) \underset{g^{-1}}{\cdot} \mathcal{R}_{\kappa\iota}\}\tau_{g} \\ &= \frac{1}{2}(\mathfrak{g}^{\kappa} \wedge \mathfrak{g}^{\iota}) \wedge \underset{g}{\star}\mathcal{R}_{\kappa\iota}\end{aligned} \tag{350}$$

Now, $\mathcal{L}_{eh}$ may be written (see, e.g., [82]) once we realize that the connection 1-forms can be written as

$$\omega^{\gamma\delta} = \frac{1}{2}\left[\mathfrak{g}^{\delta} \underset{g}{\lrcorner} d\mathfrak{g}^{\gamma} - \mathfrak{g}^{\gamma} \underset{g}{\lrcorner} d\mathfrak{g}^{\delta} + \mathfrak{g}^{\gamma} \underset{g}{\lrcorner} \left(\mathfrak{g}^{\delta} \underset{g}{\lrcorner} d\mathfrak{g}_{\alpha}\right) \mathfrak{g}^{\alpha}\right] \tag{351}$$

as

$$\mathcal{L}_{eh} = -\frac{1}{2} d\mathfrak{g}^{\alpha} \wedge \underset{g}{\star} d\mathfrak{g}_{\alpha} + \frac{1}{2} \underset{g}{\delta} \mathfrak{g}^{\alpha} \wedge \underset{g}{\star} \underset{g}{\delta} \mathfrak{g}_{\alpha} + \frac{1}{4} d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}_{\alpha} \wedge \underset{g}{\star}(d\mathfrak{g}^{\beta} \wedge \mathfrak{g}_{\beta}) - d(\mathfrak{g}^{\alpha} \wedge \underset{g}{\star} d\mathfrak{g}_{\beta}) \tag{352}$$

where d is the differential operator and $\underset{g}{\delta}$ is the Hodge coderivative operator given by a r-form field A_r by (see e.g., [82])

$$\underset{g}{\delta} A_r = (-1)^r \underset{g}{\star}^{-1} d \underset{g}{\star} A_r. \tag{353}$$

The proof of Eq.(352) is in Appendix D. Writing

$$\mathcal{L}_{g} = -\frac{1}{2} d\mathfrak{g}^{\alpha} \wedge \underset{g}{\star} d\mathfrak{g}_{\alpha} + \frac{1}{2} \underset{g}{\delta} \mathfrak{g}^{\alpha} \wedge \underset{g}{\star} \underset{g}{\delta} \mathfrak{g}_{\alpha} + \frac{1}{4} d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}_{\alpha} \wedge \underset{g}{\star}(d\mathfrak{g}^{\beta} \wedge \mathfrak{g}_{\beta}) \tag{354}$$

and taking notice of a notable identity [84]

$$-\frac{1}{2} d\mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star} d\mathfrak{g}_{\mathbf{a}} + \frac{1}{2} \underset{g}{\delta} \mathfrak{g}^{\mathbf{a}} \wedge \underset{g}{\star} \underset{g}{\delta} \mathfrak{g}_{\mathbf{a}} = -\frac{1}{2}(d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}^{\mathbf{b}}) \wedge \underset{g}{\star}(d\mathfrak{g}_{\mathbf{b}} \wedge \mathfrak{g}_{\mathbf{a}}), \tag{355}$$

we can write Eq.(354) as

$$\mathcal{L}_{g} = -\frac{1}{2}(d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}^{\mathbf{b}}) \wedge \underset{g}{\star}(d\mathfrak{g}_{\mathbf{b}} \wedge \mathfrak{g}_{\mathbf{a}}) + \frac{1}{4} d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}_{\alpha} \wedge \underset{g}{\star}(d\mathfrak{g}^{\beta} \wedge \mathfrak{g}_{\beta}) \tag{356}$$

Using Eq.(103) and Eq.(110) we can write:

$$\begin{aligned}(d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta}) \wedge \underset{g}{\star}(d\mathfrak{g}_{\alpha} \wedge \mathfrak{g}_{\beta}) &= (d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta}) \wedge \underline{\boldsymbol{h}}^{\dagger} \underset{\eta}{\star} \underline{\boldsymbol{h}}^{\clubsuit}(d\mathfrak{g}_{\alpha} \wedge \mathfrak{g}_{\beta}) \\ &= \underline{\boldsymbol{h}}^{\dagger}[\underline{\boldsymbol{h}}^{\clubsuit} d\mathfrak{g}^{\alpha} \wedge \boldsymbol{h}^{\clubsuit}\mathfrak{g}^{\beta} \wedge \underset{\eta}{\star} \underline{\boldsymbol{h}}^{\clubsuit} d\mathfrak{g}_{\alpha} \wedge \boldsymbol{h}^{\clubsuit}\mathfrak{g}_{\beta}]. \end{aligned} \tag{357}$$

$$\tag{358}$$

$$(d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}_{\alpha}) \wedge \underset{g}{\star}(d\mathfrak{g}^{\beta} \wedge \mathfrak{g}_{\beta}) = \underline{\boldsymbol{h}}^{\dagger}[\underline{\boldsymbol{h}}^{\clubsuit} d\mathfrak{g}^{\alpha} \wedge \boldsymbol{h}^{\clubsuit}\mathfrak{g}_{\alpha} \wedge \underset{\eta}{\star} \underline{\boldsymbol{h}}^{\clubsuit} d\mathfrak{g}^{\beta} \wedge \boldsymbol{h}^{\clubsuit}\mathfrak{g}_{\beta})]. \tag{359}$$

and write $\mathcal{L}_h = \underline{\boldsymbol{h}}^{\clubsuit}\mathcal{L}_g$ as

$$\mathcal{L}_{h} = \left(-\frac{1}{2}\underline{\boldsymbol{h}}^{\clubsuit}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta}) \underset{\eta^{-1}}{\cdot} \underline{\boldsymbol{h}}^{\clubsuit}(d\mathfrak{g}_{\alpha} \wedge \mathfrak{g}_{\beta}) + \frac{1}{4}\underline{\boldsymbol{h}}^{\clubsuit}(d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}_{\alpha}) \underset{\eta^{-1}}{\cdot} \underline{\boldsymbol{h}}^{\clubsuit}(d\mathfrak{g}^{\beta} \wedge \mathfrak{g}_{\beta})\right) \tau_{\eta} \tag{360}$$

Remark 6.1 A careful inspection of the Lagrangian density given by Eq.(354) reveals its remarkable structure. Indeed, the first term is of the Yang-Mills kind,

the second term may be called a gauge fixing term, since, e.g., $\underset{g}{\delta}\mathfrak{g}^{\alpha} = 0$ is analogous of the *Lorenz gauge* in electromagnetic theory. Finally the third term is an auto-interaction term, describing the coupling of the *vorticities*[50] $(d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}_{\alpha})$ of the fields $\mathfrak{g}^{\alpha}$. One way of saying that is: the distortion field $\boldsymbol{h}$ puts the Lorentz vacuum medium in motion. That motion is described by $\mathfrak{g}^{\alpha} = \boldsymbol{h}^{\dagger}(\boldsymbol{\vartheta}^{\alpha})$, saying that the deformed cosmic lattice $\boldsymbol{h}^{\dagger}(\boldsymbol{\vartheta}^{\alpha})$ is in motion relative to the ground state cosmic lattice which is defined (modulo a global Lorentz transformation) by $\{\boldsymbol{\vartheta}^{\alpha}\}$.

Remark 6.2 In the gravitational theory with the Lagrangian density given by Eq.(354) no mention of any connection on the world manifold appears. But, if some one wants to give a geometrical model interpretation for such a theory,of course, the simple one is to say that the gravitational field generates a teleparallel geometry with a metric compatible connection defined on the world manifold by[51] $\nabla^{-}_{\mathfrak{e}_{\alpha}}\mathfrak{g}^{\beta} = 0$, $\nabla^{-}\boldsymbol{g} = 0$. For that connection the Riemann curvature tensor is null the whereas the torsion 2-form fields are given by $\Theta^{\alpha} = d\mathfrak{g}^{\alpha}$ (i.e., the gravitational field is torsion in this model). Only on a second thought someone would think to introduce a Levi-Civita connection D on the world manifold such that $D^{-}_{\mathfrak{e}_{\alpha}}\mathfrak{g}^{\beta} = -L^{\beta}_{\alpha\gamma}\mathfrak{g}^{\gamma}$, $D^{-}\boldsymbol{g} = 0$, for which the torsion 2-forms $\Theta^{\alpha} = d\mathfrak{g}^{\alpha} + \omega^{\alpha}_{\beta} \wedge \mathfrak{g}^{\beta} = 0$ and the curvature 2-forms $\mathcal{R}^{\alpha}_{\beta} = d\omega^{\alpha}_{\beta} + \omega^{\alpha}_{\beta} \wedge \omega^{\alpha}_{\beta} \neq 0$. However, since history did not follow that path, it was the geometry associated to the Levi-Civita connection that was first discovered by Einstein and Grossmann. More on this issue in [70].

Remark 6.3 Eq.(360) express $\mathcal{L}_h$ as a functional of $\boldsymbol{h}^{\clubsuit}$ and uses only objects belonging to the Minkowski spacetime structure. However, to perform the variation of $\mathcal{L}_h$ in order to determine the equations of motion for the extensor field $\boldsymbol{h}^{\clubsuit}$ is an almost impracticable task, which may leads one to appreciate the tricks used in Section 5 to derive the equation of motion for that extensor field $\boldsymbol{h}$. The derivation of the equations of motion directly for the potential fields $\mathfrak{g}^{\mathbf{a}}$ is a more easy (but still involved) task and is given in Appendix E.

6.1 The Total Lagrangian Density for the Massive Gravitational Field Plus the Matter Fields

In this section as we study the interaction of the gravitational potentials $\mathfrak{g}^{\mathbf{a}}$ with the matter fields described by a Lagrangian density $\mathcal{L}_m$. Moreover, as before in Section 5.1.2, we add a term corresponding to the Lorentz vacuum energy density, the one given in terms of the cosmological constant, which we prefer

[50]The name vorticity has been used here by the following reason. In *GRT*, as well known (see, e.g., [82, 86] a reference frame in $(M, \boldsymbol{g}, D, \tau_{\boldsymbol{g}}$ is a timelike vector field Z (pointing to the future). The physically equivalent 1-form $\alpha = \boldsymbol{g}(\ ,Z)$ has the unique decomposition $D\alpha = \mathbf{a} \otimes \alpha + \omega + \sigma + \frac{1}{3}\mathfrak{E}\mathbf{h}$,where $\mathbf{h} \in \sec T^{0}_{2}M$ is the projection tensor $\mathbf{h} = \boldsymbol{g} - \alpha \otimes \alpha$, $\mathbf{a}$ *acceleration* of Z, ω is the *rotation* tensor *(* or *vortex)* of Z, σ is the *shear* of Z and $\mathfrak{E}$ is the *expansion ratio* of Z. We can show that the non null components of ω are the same as the ones in $\alpha \wedge d\alpha$.

[51]The $\{\mathfrak{e}_{\alpha}\}$ is the dual basis of $\{\mathfrak{g}^{\alpha}\}$.

here to write as a 'graviton mass' term, i.e., we have:

$$\mathcal{L}' = \mathcal{L}_{eh} + \frac{1}{2}m^2\mathfrak{g}_\alpha \wedge \underset{g}{\star}\mathfrak{g}^\alpha + \mathcal{L}_m \tag{361}$$

or equivalently, since $\mathcal{L}_{eh}$ and $\mathcal{L}_g$ differs from an exact differential. we use

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_g + \frac{1}{2}m^2\mathfrak{g}_\alpha \wedge \underset{g}{\star}\mathfrak{g}^\alpha + \mathcal{L}_m \\ &= \mathcal{L}'_g + \mathcal{L}_m\end{aligned}$$

The resulting equations of motion (whose derivation for completeness are given in the Appendix E) may be written putting

$$\star\, t^\alpha = \frac{\partial \mathcal{L}'_g}{\partial \mathfrak{g}_\alpha},\ \underset{g}{\star}\,\mathcal{S}^\alpha = \frac{\partial \mathcal{L}'_g}{\partial d\mathfrak{g}_\alpha},\ -\underset{g}{\star}\mathcal{T}^\alpha = \underset{g}{\star}T^\alpha := -\frac{\partial \mathcal{L}_m}{\partial \mathfrak{g}_\alpha}. \tag{362}$$

$$\boxed{-\underset{g}{\star}\,G^\alpha = d\underset{g}{\star}\,\mathcal{S}^\alpha + \underset{g}{\star}\mathfrak{t}^\alpha = -\underset{g}{\star}\,\mathcal{T}^\alpha,} \tag{363}$$

where $\underset{g}{\star}G^\alpha \in \sec \bigwedge^3 T^*U$ are the Einstein 3-form fields while $\underset{g}{\star}\mathfrak{t}^\kappa \in \sec \bigwedge^3 T^*U$ and $\underset{g}{\star}\mathcal{S}^\alpha \in \sec \bigwedge^2 T^*U$ are given by (recall Eq.(559) and Eq.(560) of Appendix E):

$$\underset{g}{\star}\,\mathfrak{t}^\kappa = \frac{\partial \mathcal{L}'_g}{\partial \mathfrak{g}_\kappa} = \frac{\partial \mathcal{L}_g}{\partial \mathfrak{g}_\kappa} + m^2\mathfrak{g}^\kappa = \mathfrak{g}_\kappa \underset{g}{\lrcorner} \mathcal{L}_g - (\mathfrak{g}_\kappa \underset{g}{\lrcorner} d\mathfrak{g}_\alpha) \wedge \underset{g}{\star}\mathcal{S}^\alpha + m^2\mathfrak{g}^\kappa \tag{364}$$

$$\underset{g}{\star}\,\mathcal{S}^\kappa = \frac{\partial \mathcal{L}_g}{\partial d\mathfrak{g}_\kappa} = -\mathfrak{g}^\alpha \wedge \underset{g}{\star}(d\mathfrak{g}_\alpha \wedge \mathfrak{g}_\kappa) + \frac{1}{2}\mathfrak{g}_\kappa \wedge \underset{g}{\star}(d\mathfrak{g}^\alpha \wedge \mathfrak{g}_\alpha). \tag{365}$$

We write moreover

$$\underset{g}{\star}\mathcal{S}^\kappa = \underset{g}{\star}d\mathfrak{g}^\kappa + \underset{g}{\star}\mathfrak{K}^\kappa, \tag{366}$$

$$\underset{g}{\star}\mathfrak{K}^\kappa = -(\mathfrak{g}^\kappa \underset{g}{\lrcorner} \underset{g}{\star}\,\mathfrak{g}^\alpha) \wedge \underset{g}{\star}d\underset{g}{\star}\,\mathfrak{g}_\alpha + \frac{1}{2}\mathfrak{g}^\kappa \wedge \underset{g}{\star}(d\mathfrak{g}^\alpha \wedge \mathfrak{g}_\alpha) \tag{367}$$

and insert this result in Eq.(363) obtaining:

$$d\underset{g}{\star}\,d\mathfrak{g}^\kappa + m^2\underset{g}{\star}\,\mathfrak{g}^\kappa = -\underset{g}{\star}\left(\mathfrak{t}^\kappa + \mathcal{T}^\kappa + \underset{g}{\star}^{-1}d\underset{g}{\star}\,\mathfrak{K}^\kappa\right) \tag{368}$$

Applying the operator $\underset{g}{\star}^{-1}$ to both sides of that equation, we get

$$-\underset{g}{\delta}d\mathfrak{g}^\kappa + m^2\mathfrak{g}^\kappa = -\left(t^\kappa + \mathcal{T}^\kappa + \underset{g}{\delta}\mathfrak{K}^\kappa\right) \tag{369}$$

Calling

$$\mathfrak{F}^\kappa := d\mathfrak{g}^\kappa$$

we have he following *Maxwell like* equations for the gravitational fields $\mathfrak{F}^{\kappa}$:

$$d\mathfrak{F}^{\kappa} = 0, \tag{370a}$$
$$\underset{g}{\delta}\mathfrak{F}^{\kappa} = \left(t^{\kappa} + \mathfrak{T}^{\kappa} + m^2\mathfrak{g}^{\kappa}\right), \tag{370b}$$
$$\mathfrak{T}^{\kappa} = \mathcal{T}^{\kappa} + \underset{g}{\delta}\mathfrak{K}^{\kappa} \tag{370c}$$

However, take notice that the above system of differential equations is non linear in the gravitational potentials $\mathfrak{g}^{\kappa}$.

We now may define the total energy-momentum of the system consisting of the matter plus the gravitational field either as or $\mathbf{P} = P_{\kappa}\vartheta^{\kappa}$

$$P_{\kappa} := -\int_{U} \underset{g}{\star}(\mathfrak{t}_{\kappa} + \mathfrak{T}^{\kappa} + m^2\mathfrak{g}_{\kappa}) = \int_{\partial U} \underset{g}{\star} d\mathfrak{g}_{\kappa}. \tag{371}$$

or $\mathbf{P}' = P'_{\kappa}\vartheta^{\kappa}$

$$P'_{\kappa} := -\int_{U} \underset{g}{\star}(\mathfrak{t}_{\kappa} + \mathcal{T}_{\kappa} + m^2\mathfrak{g}_{\kappa}) = \int_{\partial U} \underset{g}{\star}\mathcal{S}_{\kappa}, \tag{372}$$

6.1.1 Energy-Momentum Conservation Law

The Maxwell like formulation (Eqs. (370)) or Eq.(363) immediately imply two possible energy-momentum conservation laws in our theory (i.e., both the P_{κ} : as well as the P'_{κ} are conserved), but in order to avoid any misunderstanding some remarks are necessary.

Remark 6.4 The values of P_{κ} (Eq.(371)) as well as P'_{κ} (Eq.(372)) are, of course, independent of the coordinate system used to calculate them. This is also the case in *GRT* where identical equations hold for models with global $\boldsymbol{g}$-orthonormal cotetrad fields. However, in *GRT* expressions anlagous to $\mathbf{P} = P_{\kappa}\vartheta^{\kappa}$ or $\mathbf{P}' = P'_{\kappa}\vartheta^{\kappa}$ can be given meaning only for asymptotically flat spacetimes. Also, it is obvious that in our theory as well in *GRT* both P_{κ} or P'_{κ} depend on the choice of the cotetrad field $\{\mathfrak{g}^{\alpha}\}$. The difference is that in our theory the $\{\mathfrak{g}^{\alpha}\}$, the gravitational potentials (defined by the extensor field $\boldsymbol{h}$ modulus an arbitrary local Lorentz rotation Λ which is hidden in the definition of $\boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h} = \boldsymbol{h}^{\dagger}\Lambda^{\dagger}\eta\Lambda\boldsymbol{h}$) are by chance a section of the $\mathbf{g}$-orthonormal frame bundle of an effective Lorentzian spacetime (with the $\underset{g}{\star}t_{\kappa}$ being thus gauge dependent objects) whereas in *GRT* the $\{\mathfrak{g}^{\alpha}\}$ do not have any interpretation different from the one of being a section of the $\mathbf{g}$-orthonormal frame bundle of theLorentzian spacetime representing a particular gravitational field. Moreover in *GRT* we can write equations similar to Eq.(364) also for coordinate basis of the linear frame bundle of the Lorentzian spacetime, thus obtaining yet different 'energy-mometum pseudo-tensors', for which the *values* of analogous integral to Eq.(372) depends on the coordinate system used for their calculations, something that is clearly a nonsense [6, 69].

Remark 6.5 The condition of possessing a global $\boldsymbol{g}$-orthonormal tetrad field is, of course, not satisfied by a general Lorentzian structure[52] $(M, \boldsymbol{g})$ and if we want that *GRT* admits a total energy-momentum conservation law for the system consisting of the matter and gravitational fields, such a condition must be added as an *extra* condition for the spacetime structure, a fact that we already mentioned in Section 1.

Remark 6.6 Another very well known fact concerning the t^{α} is that their components are not symmetric. Then, for example in [98] it is claimed that in order to obtain an angular momentum conservation law it is necessary to use superpotentials different from the $\mathcal{S}^{\alpha}$ which produce a symmetric t^{α}. The choice made, e.g., in [98] is then to use the so-called Landau-Lifschitz pseudo-energy momentum tensor [54], which as well known is a symmetric object[53]. However, we will show next that with the original non symmetric t^{α} we can get a total angular momentum conservation law which includes the spin density of the gravitational field.

6.1.2 Angular Momentum Conservation Law

Consider the global chart for $\mathbb{R}^4$ with coordinates in the Einstein-Lorentz-Poincaré gauge $\{\mathrm{x}^{\mu}\}$ and as previously put $\boldsymbol{\vartheta}^{\mu} = d\mathrm{x}^{\mu}$. Calling

$$\underset{g}{\star}\mathbf{T}^{\alpha} := \underset{g}{\star}(t^{\kappa} + \mathfrak{T}^{\kappa} + +m^2\mathfrak{g}^{\kappa}) \tag{373}$$

the (non symmetric) *total* energy momentum 3-forms of matter plus the gravitational field, we define the *chart dependent* density of *total orbital angular momentum* of the matter as the 3-form fields

$$\underset{g}{\star}\mathbf{L}_m^{\alpha\beta} := \underline{\boldsymbol{h}}^{\dagger}\{\boldsymbol{h}^{\clubsuit}(\mathrm{x}^{\alpha}) \wedge \underset{\eta}{\star}\boldsymbol{h}^{\clubsuit}\mathbf{T}^{\beta} - \boldsymbol{h}^{\clubsuit}(\mathrm{x}^{\beta}) \wedge \underset{\eta}{\star}\boldsymbol{h}^{\clubsuit}\mathbf{T}^{\alpha}\}, \tag{374}$$

which can be written as

$$\underset{g}{\star}\mathbf{L}_m^{\alpha\beta} = \mathrm{x}^{\alpha}\underset{g}{\star}\mathbf{T}^{\beta} - \mathrm{x}^{\beta}\underset{g}{\star}\mathbf{T}^{\alpha}. \tag{375}$$

Of course,

$$d\underset{g}{\star}\mathbf{L}_m^{\alpha\beta} = \boldsymbol{\vartheta}^{\alpha} \wedge \underset{g}{\star}\mathbf{T}^{\beta} - \boldsymbol{\vartheta}^{\beta} \wedge \underset{g}{\star}\mathbf{T}^{\alpha} \neq 0. \tag{376}$$

However, let us define the density (3-forms) of *orbital* angular momentum of the gravitational field by

$$\underset{g}{\star}\mathbf{L}_g^{\alpha\beta} := \underline{\boldsymbol{h}}^{\dagger}\{\mathfrak{g}^{\alpha} \wedge \underset{\eta}{\star}\boldsymbol{h}^{\clubsuit}\mathfrak{F}^{\beta} - \mathfrak{g}^{\beta} \wedge \underset{\eta}{\star}\boldsymbol{h}^{\clubsuit}\mathfrak{F}^{\alpha}\}, \tag{377}$$

which can be written as

$$\underset{g}{\star}\mathbf{L}_g^{\alpha\beta} = \boldsymbol{\vartheta}^{\alpha} \wedge \underset{g}{\star}\mathfrak{F}^{\beta} - \boldsymbol{\vartheta}^{\beta} \wedge \underset{g}{\star}\mathfrak{F}^{\alpha}, \tag{378}$$

[52]Such a condition, as well known [37, 82] is a necessary one for existence of spinor fields
[53]For more details on those issues see [69].

and the density of *total orbital angular momentum* of the matter plus the gravitational field by

$$\underset{g}{\star}\mathbf{L}_t^{\alpha\beta} = \underset{g}{\star}\mathbf{L}_m^{\alpha\beta} + \underset{g}{\star}\mathbf{L}_g^{\alpha\beta}. \tag{379}$$

Then we immediately get that

$$\begin{aligned} d\underset{g}{\star}\mathbf{L}_t^{\alpha\beta} &= \boldsymbol{\vartheta}^\alpha \wedge \underset{g}{\star}\mathbf{T}^\beta - \boldsymbol{\vartheta}^\beta \wedge \underset{g}{\star}\mathbf{T}^\alpha - \boldsymbol{\vartheta}^\alpha \wedge d\underset{g}{\star}\mathfrak{F}^\beta + \boldsymbol{\vartheta}^\beta \wedge d\underset{g}{\star}\mathfrak{F}^\alpha \\ &= \boldsymbol{\vartheta}^\alpha \wedge \underset{g}{\star}\mathbf{T}^\beta - \boldsymbol{\vartheta}^\beta \wedge \underset{g}{\star}\mathbf{T}^\alpha - \boldsymbol{\vartheta}^\alpha \wedge \underset{g}{\star}\mathbf{T}^\beta + \boldsymbol{\vartheta}^\beta \wedge \underset{g}{\star}\mathbf{T}^\alpha \\ &= 0. \end{aligned} \tag{380}$$

However, Eq.(380) does not contains yet the spin angular momentum of matter and the spin angular momentum of the gravitational field and thus cannot be considered as a satisfactory equation for the conservation of total angular momentum. To go on, we proceed as follows. First we write the total Lagrangian density for the gravitational field as

$$\mathcal{L} = \mathcal{L}_{eh} + \mathcal{L}_m, \tag{381}$$

where $\mathcal{L}_{eh}$ is defined by Eq.(350), i.e.,

$$\mathcal{L}_{eh} = \frac{1}{2}\mathcal{R}_{\kappa\iota} \wedge \underset{g}{\star}(\mathfrak{g}^\kappa \wedge \mathfrak{g}^\iota), \tag{382}$$

but where now we suppose that $\mathcal{L}_{eh}$ is a functional of the gravitational potentials $\mathfrak{g}^\kappa$ and the connection 1-form fields $\omega_{\kappa\iota}$ appearing in $\mathcal{R}_{\kappa\iota}$ which we take as independent variables to start. Moreover, we suppose that $\mathcal{L}_m$ depends on the field variables ϕ^A (which in general are indexed form fields[54]) and their exterior covariant derivatives, i.e., it is a functional of the kind $\mathcal{L}_m = \mathcal{L}_m(\phi^A, d\phi^A, \omega_{\kappa\iota})$.

If we make the variation of the action $\int_U(\mathcal{L}_{eh} + \mathcal{L}_m)$ with respect to $\omega_{\kappa\iota}$ we get

$$\begin{aligned} &\boldsymbol{\delta}\int_U(\mathcal{L}_{eh} + \mathcal{L}_m) \\ &= \int_U(\boldsymbol{\delta}\omega_{\kappa\iota} \wedge \frac{\partial\mathcal{L}_{eh}}{\partial\omega_{\kappa\iota}} + \boldsymbol{\delta}d\omega_{\kappa\iota} \wedge \frac{\partial\mathcal{L}_{eh}}{\partial d\omega_{\kappa\iota}} + \boldsymbol{\delta}\omega_{\kappa\iota} \wedge \frac{\partial\mathcal{L}_m}{\partial\omega_{\kappa\iota}}) \\ &= \int_U(\boldsymbol{\delta}\omega_{\kappa\iota} \wedge \left[\frac{\partial\mathcal{L}_{eh}}{\partial\omega_{\kappa\iota}} + d\left(\frac{\partial\mathcal{L}_{eh}}{\partial d\omega_{\kappa\iota}}\right) + \frac{\partial\mathcal{L}_m}{\partial\omega_{\kappa\iota}}\right] + d\left(\boldsymbol{\delta}\omega_{\kappa\iota} \wedge \frac{\partial\mathcal{L}_{eh}}{\partial d\omega_{\kappa\iota}}\right), \end{aligned} \tag{383}$$

from where we obtain from the principle of stationary action and with the usual hypothesis that the variations vanishes on the boundary ∂U of U that

$$\frac{\partial\mathcal{L}_{eh}}{\partial\omega_{\kappa\iota}} + d\left(\frac{\partial\mathcal{L}_{eh}}{\partial d\omega_{\kappa\iota}}\right) + \frac{\partial\mathcal{L}_m}{\partial\omega_{\kappa\iota}} = 0 \tag{384}$$

[54]Even spinor fields may be represented in he Clifford bundle formalism as some equivlaence classes of non homogenous form fields. See, e.g., [82] for details.

In metric affine theories the spin [2, 45] (density) angular momentum of matter is defined by

$$\underset{g}{\star}\mathbf{S}_m^{\kappa\iota} = \frac{\partial \mathcal{L}_m}{\partial \omega_{\kappa\iota}}. \tag{385}$$

and we will accept it as a good one[55]. Now,

$$\begin{aligned}
\frac{\partial \mathcal{L}_{eh}}{\partial \omega_{\kappa\iota}} &= \frac{1}{2}\left(\omega_\alpha^\kappa \wedge \underset{g}{\star}(\mathfrak{g}^\alpha \wedge \mathfrak{g}^\iota) + \omega_\alpha^\iota \wedge \underset{g}{\star}(\mathfrak{g}^\kappa \wedge \mathfrak{g}^\alpha)\right), \\
d\left(\frac{\partial \mathcal{L}_{eh}}{\partial d\omega_{\kappa\iota}}\right) &= \frac{1}{2}\underset{g}{\star}(\mathfrak{g}^\kappa \wedge \mathfrak{g}^\mathbf{l}).
\end{aligned} \tag{386}$$

Thus, defining

$$\underset{g}{\star}\mathbf{S}_g^{\kappa\iota} := \omega_\alpha^\kappa \wedge \underset{g}{\star}(\mathfrak{g}^\alpha \wedge \mathfrak{g}^\iota) + \omega_\alpha^\iota \wedge \underset{g}{\star}(\mathfrak{g}^\kappa \wedge \mathfrak{g}^\alpha) \tag{387}$$

as the spin (density) angular momentum of the gravitational field, we get

$$d\underset{g}{\star}(\mathfrak{g}^\kappa \wedge \mathfrak{g}^\mathbf{l}) = -2\underset{g}{\star}\left(\mathbf{S}_m^{\kappa\iota} + \mathbf{S}_g^{\kappa\iota}\right), \tag{388}$$

and thus[56]

$$d\underset{g}{\star}\left(\mathbf{S}_m^{\kappa\iota} + \mathbf{S}_g^{\kappa\iota}\right) = 0. \tag{389}$$

We now define the total angular momentum of the matter field plus the gravitational field as

$$\underset{g}{\star}\mathbf{J}^{\alpha\beta} := \underset{g}{\star}(\mathbf{L}_m^{\alpha\beta} + \mathbf{S}_m^{\kappa\iota}) + \underset{g}{\star}(\mathbf{L}_g^{\alpha\beta} + +\mathbf{S}_g^{\kappa\iota}), \tag{390}$$

and due Eq.(380) and Eq.(389) we have

$$d\underset{g}{\star}\mathbf{J}^{\alpha\beta} = 0. \tag{391}$$

Eq.(391) express the law of conservation of total angular momentum in our theory, and surprisiling it seems that the orbital and spin angualr mometum are separately conserved.

6.1.3 Wave Equation for the $\mathfrak{g}^\kappa$

Wave equation for the gravitational potential $\mathfrak{g}^\kappa$ can easily be written in the *effective* Lorentzian spacetime structure $(M, \mathbf{g}, D, \tau_g, \uparrow)$ already introduced. Indeed, to get that wave equation we add the term $d\underset{g}{\delta}\mathfrak{g}^\kappa$ to both members Eq.(369) getting

$$-\underset{g}{\delta}d\mathfrak{g}^\kappa - d\underset{g}{\delta}\mathfrak{g}^\kappa + m^2\mathfrak{g}^\kappa = -\left(\mathfrak{t}^\kappa + \mathcal{T}^\kappa + \underset{g}{\delta}\mathfrak{K}^\kappa + d\underset{g}{\delta}\mathfrak{g}^\kappa\right). \tag{392}$$

[55] In particular in **??** it is shown that this agree with spin density which has as source the antisymmetric part of the canonical energy-mometnum tensor of the *matter* field.

[56] An equations anlogous to this one has been found originally by Bramsom [7].

We now recall the definition of the Hodge D'Alembertian, which in the Clifford bundle formalism is the square of the Dirac operator ($\boldsymbol{\partial} := \mathfrak{g}^{\kappa} \underset{g}{} D^{-}_{\mathfrak{g}_{\kappa}} = d - \underset{g}{\delta}$) acting on multiform fields [82], i.e., we can write

$$\mathring{\Diamond}\mathfrak{g}^{\kappa} := \boldsymbol{\partial}^2 \mathfrak{g}^{\kappa} = (-\underset{g}{\delta} d - d\underset{g}{\delta})\mathfrak{g}^{\kappa}. \tag{393}$$

We recall moreover the following nontrivial decomposition [82] of $\boldsymbol{\partial}^2$,

$$\boldsymbol{\partial}^2 \mathfrak{g}^{\kappa} = \boldsymbol{\partial} \underset{g}{\cdot} \boldsymbol{\partial}\mathfrak{g}^{\kappa} + \boldsymbol{\partial} \wedge \boldsymbol{\partial}\mathfrak{g}^{\kappa}, \tag{394}$$

where $\square := \boldsymbol{\partial} \underset{g}{\cdot} \boldsymbol{\partial}$ is the *covariant D'Alembertian* and $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ is the *Ricci operator associated to the Levi-Civita connection of* $\boldsymbol{g}$. If

$$D^{-}_{\mathfrak{g}_{\alpha}} \mathfrak{g}^{\beta} = -L^{\beta}_{\alpha\rho}\mathfrak{g}_{\rho}, \tag{395}$$

then

$$\begin{array}{lrcl} \text{(a)} & \boldsymbol{\partial} \underset{g}{\cdot} \boldsymbol{\partial} & = & \eta^{\alpha\beta}(D^{-}_{\mathfrak{g}_{\alpha}} D^{-}_{\mathfrak{g}_{\beta}} - L^{\rho}_{\alpha\beta} D^{-}_{\mathfrak{g}_{\rho}}) \\ \text{(b)} & \boldsymbol{\partial} \wedge \boldsymbol{\partial} & = & \mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta}(D^{-}_{\mathfrak{g}_{\alpha}} D^{-}_{\mathfrak{g}_{\beta}} - L^{\rho}_{\alpha\beta} D^{-}_{\mathfrak{g}_{\rho}}), \end{array} \tag{396}$$

and an easily computation shows that

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial}\mathfrak{g}^{\kappa} = \mathcal{R}^{\kappa}, \tag{397}$$

where $\mathcal{R}^{\kappa} : U \to \bigwedge^1 U$ are the Ricci 1-form fields, given by:

$$\mathcal{R}^{\kappa} = R^{\kappa}_{\iota}\mathfrak{g}^{\iota}, \tag{398}$$

where R^{κ}_{ι} are the components of the Ricci tensor in the basis defined by $\{\mathfrak{g}^{\alpha}\}$. Then we have the following wave equations for the gravitational potentials:

$$\square\mathfrak{g}^{\kappa} + m^2\mathfrak{g}^{\kappa} = -(\mathcal{T}^{\kappa} + \mathfrak{t}^{\kappa} + \underset{g}{\delta}\mathfrak{K}^{\kappa} + d\underset{g}{\delta}\mathfrak{g}^{\kappa}). \tag{399}$$

7 Hamiltonian Formalism

7.1 The Hamiltonian 3-form Density $\mathcal{H}$

We start with the Lagrangian density[57] given by Eq.(354), i.e.,

$$\mathcal{L}_g(\mathfrak{g}^{\alpha}, d\mathfrak{g}^{\alpha}) = -\frac{1}{2} d\mathfrak{g}^{\alpha} \wedge \underset{g}{\star} d\mathfrak{g}_{\alpha} + \frac{1}{2}\underset{g}{\delta}\mathfrak{g}^{\alpha} \wedge \underset{g}{\star}\underset{g}{\delta}\mathfrak{g}_{\alpha} + \frac{1}{4} d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}_{\alpha} \wedge \underset{g}{\star}(d\mathfrak{g}^{\beta} \wedge \mathfrak{g}_{\beta}), \tag{400}$$

[57] A rigorous formulation of the Lagrangian and Hamiltonian formalism in field theory needs at least the introduction of the concepts of jet bundles and the Legendre bundles. Such a theory is finely presented, e.g., in the excellent texts [38, 52]. Here we give a formulation of the theory without mentioning those concepts, but which is very similar to the standard one used in physicists books on field theory, and which for the best of our knowledge has been first used in [97] and developed with mastery in [103] (see also [53]). Also, we recall that the Hamiltonian formalism for *GRT* as usually presented on some textbooks has been introduced in [1], where the concept of *ADM* energy first appeared. See Section 7.3.

and as usual in the Lagrangian formalism we define the conjugate momenta $\mathfrak{p}_\alpha \in \sec \bigwedge^3 T^*M \hookrightarrow \mathcal{C}\ell(\mathfrak{g}, M)$ to the potentials $\mathfrak{g}^\alpha$ by:

$$\mathfrak{p}_\alpha = \frac{\partial \mathcal{L}_g}{\partial d\mathfrak{g}^\alpha}, \tag{401}$$

and before proceeding we recall that according to Eq.(365), it is:

$$\mathfrak{p}_\alpha \equiv \underset{g}{\star} \mathcal{S}_\alpha. \tag{402}$$

Next, supposing that we can solve Eq.(401) for the $d\mathfrak{g}^\alpha$ as functions of the $\mathfrak{p}_\alpha$, we introduce a *Legendre* transformation with respect to the fields $d\mathfrak{g}^\alpha$ by

$$\mathbf{L} : (\mathfrak{g}^\alpha, \mathfrak{p}_\alpha) \mapsto \mathbf{L}(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha) = d\mathfrak{g}^\alpha \wedge p_\alpha - \mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha(\mathfrak{p}_\alpha)) \tag{403}$$

We write in what follows

$$\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha) := \mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha(\mathfrak{p}_\alpha)) \tag{404}$$

We observe that

$$\begin{aligned}\boldsymbol{\delta}\mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha) &= d(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\partial \mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha)}{\partial d\mathfrak{g}^\alpha}) + \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \left[\frac{\partial \mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha)}{\partial \mathfrak{g}^\alpha} - d(\frac{\partial \mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha)}{\partial d\mathfrak{g}^\alpha})\right] \\ &= d(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) + \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha)}{\boldsymbol{\delta}\mathfrak{g}^\alpha}.\end{aligned} \tag{405}$$

Also from Eq.(403) we immediately have:

$$\begin{aligned}\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha) &= \boldsymbol{\delta}(d\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) - \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\partial \mathbf{L}}{\partial \mathfrak{g}^\alpha} - \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \frac{\partial \mathbf{L}}{\partial \mathfrak{p}_\alpha} \\ &= d(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) + \boldsymbol{\delta}\mathfrak{g}^\alpha \left(-d\mathfrak{p}_\alpha - \frac{\partial \mathbf{L}}{\partial \mathfrak{g}^\alpha}\right) + \boldsymbol{\delta}(d\mathfrak{g}^\alpha - \frac{\partial \mathbf{L}}{\partial \mathfrak{p}_\alpha}) \wedge \mathfrak{p}_\alpha \\ &= d(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) + \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \left(\frac{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha)}{\boldsymbol{\delta}\mathfrak{g}^\alpha}\right) + \left(\frac{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha)}{\boldsymbol{\delta}\mathfrak{p}_\alpha}\right) \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha,\end{aligned} \tag{406}$$

where we put:

$$\begin{aligned}\frac{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha)}{\boldsymbol{\delta}\mathfrak{g}^\alpha} &:= -d\mathfrak{p}_\alpha - \frac{\partial \mathbf{L}}{\partial \mathfrak{g}^\alpha}, \\ \frac{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha)}{\boldsymbol{\delta}\mathfrak{p}_\alpha} &:= d\mathfrak{g}^\alpha - \frac{\partial \mathbf{L}}{\partial \mathfrak{p}_\alpha}.\end{aligned} \tag{407}$$

From Eqs. (405) and (406) we have

$$\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha)}{\boldsymbol{\delta}\mathfrak{g}^\alpha} = \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \left(\frac{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha)}{\boldsymbol{\delta}\mathfrak{g}^\alpha}\right) + \left(\frac{\boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha)}{\boldsymbol{\delta}\mathfrak{p}_\alpha}\right) \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha. \tag{408}$$

To define the Hamiltonian form we need something to do the role of time for our manifold, and we choose this time to be given by the flow of an arbitrary timelike vector field[58] $\mathbf{Z} \in \sec TM$ such that $\boldsymbol{g}(\mathbf{Z},\mathbf{Z}) = 1$. Moreover we define $Z = \boldsymbol{g}(\mathbf{Z},) \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(\mathtt{g}, M)$. With this choice the variation $\boldsymbol{\delta}$ is generated by the Lie derivative $\pounds_{\mathbf{Z}}$. Using Cartan's 'magical formula' (see, e.g., [82]) we have

$$\boldsymbol{\delta}\mathfrak{L}_g = \pounds_{\mathbf{Z}}\mathfrak{L}_g = d(Z\lrcorner\mathfrak{L}_g) + Z\lrcorner d\mathfrak{L}_g = d(Z\lrcorner\mathfrak{L}_g). \tag{409}$$

Then, using Eq.(406) we can write

$$d(Z\lrcorner\mathfrak{L}_g) = d(\pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) + \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} + \pounds_{\mathbf{Z}}\mathfrak{p}_\alpha \wedge \left(\frac{\boldsymbol{\delta}\mathfrak{L}}{\boldsymbol{\delta}\mathfrak{p}_\alpha}\right) \tag{410}$$

and taking into account Eq.(408) we get

$$d(\pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha - Z\lrcorner\mathfrak{L}_g) = \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha}. \tag{411}$$

Now, we define the *Hamiltonian* 3-form by

$$\mathcal{H}(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha) :=\pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha - Z\lrcorner\mathfrak{L}_g. \tag{412}$$

We immediately have in view of Eq.(411) that when the field equations for the *free* gravitational field are satisfied (i.e., when the Euler-Lagrange functional is null, $\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} = 0$) that

$$d\mathcal{H} = 0. \tag{413}$$

Thus $\mathcal{H}$ is a *conserved* Noether current. We next write

$$\mathcal{H} = Z^\alpha\mathcal{H}_\alpha + dB \tag{414}$$

and find some equivalent expressions for $\mathcal{H}_\alpha$. To start, we rewrite Eq.(412) taking into account Eq.(401), Cartan's magical formula and some Clifford algebra identities as

$$\begin{aligned}\mathcal{H} &= -Z\lrcorner\mathfrak{L}_g + \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha \\ &= -Z\lrcorner\mathfrak{L}_g + d(Z\lrcorner\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha + (Z\lrcorner d\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha \\ &- Z\lrcorner\mathfrak{L}_g + d\left((Z\lrcorner\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha\right) - (Z\lrcorner\mathfrak{g}^\alpha) \wedge d\mathfrak{p}_\alpha + (Z\lrcorner d\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha \\ &= Z^\alpha\left(\mathfrak{g}_\alpha\lrcorner\mathfrak{L}_g + (\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa) \wedge \mathfrak{p}_\kappa - d\mathfrak{p}_\alpha\right) + d\left(Z^\alpha\mathfrak{p}_\alpha\right),\end{aligned} \tag{415}$$

from where we get

$$\begin{gathered}\boxed{\mathcal{H}_\alpha = \mathfrak{g}_\alpha\lrcorner\mathfrak{L}_g + (\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa) \wedge \mathfrak{p}_\kappa - d\mathfrak{p}_\alpha,} \\ \boxed{B = Z^\alpha\mathfrak{p}_\alpha.}\end{gathered} \tag{416}$$

[58] An arbitrary timelike vector field $\mathbf{Z} \in \sec TM$ such that $\boldsymbol{g}(\mathbf{Z},\mathbf{Z}) = 1$ defines in *GRT* a *reference frame*, whose integral lines represent *observers*. For details, consult, e.g., [82, 86].

Next we utilize the definition of the Legendre transform (Eq.(403)), and some Clifford algebra identities to write Eq.(412) as :

$$\begin{aligned}\mathcal{H} &= -Z\lrcorner\mathfrak{L}_g + d(Z\lrcorner\mathfrak{g}^\alpha)\wedge\mathfrak{p}_\alpha + (Z\lrcorner d\mathfrak{g}^\alpha)\wedge\mathfrak{p}_\alpha \\ &= Z\lrcorner\mathbf{L} - Z\lrcorner(\mathfrak{g}^\kappa\wedge\mathfrak{p}_\kappa) + d(Z\lrcorner\mathfrak{g}^\kappa)\wedge\mathfrak{p}_\kappa + (Z\lrcorner d\mathfrak{g}^\kappa)\wedge\mathfrak{p}_\kappa \\ &= Z^\alpha\left(\mathbf{L}_\alpha + (\mathfrak{g}_\alpha\lrcorner\mathfrak{p}_\kappa)\wedge d\mathfrak{g}^\kappa - (\mathfrak{g}_\alpha\lrcorner\mathfrak{g}^\kappa)d\mathfrak{p}_\kappa\right) + d\left(Z^\alpha\mathfrak{p}_\alpha\right),\end{aligned} \tag{417}$$

where we put $\mathbf{L}_\alpha = \mathfrak{g}_\alpha\lrcorner\mathbf{L}$. Thus we also have

$$\begin{aligned}&\boxed{\mathcal{H}_\alpha = \mathbf{L}_\alpha + (\mathfrak{g}_\alpha\lrcorner\mathfrak{p}_\kappa)\wedge d\mathfrak{g}^\kappa - d\mathfrak{p}_\alpha,} \\ &\boxed{B = Z^\alpha\mathfrak{p}_\alpha.}\end{aligned} \tag{418}$$

Now, recalling Eq.(411), we can write

$$d(Z^\alpha\mathcal{H}_\alpha + dB) + \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} = 0. \tag{419}$$

Then,

$$\begin{aligned}&dZ^\alpha\wedge\mathcal{H}_\alpha + Z^\alpha d\mathcal{H}_\alpha + d(Z\lrcorner\mathfrak{g}^\alpha)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} + (Z\lrcorner d\mathfrak{g}^\alpha)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} \\ &= dZ^\alpha\wedge\left(\mathcal{H}_\alpha + \frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha}\right) + Z^\alpha\left(d\mathcal{H}_\alpha + d(\mathfrak{g}_\alpha\lrcorner\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} + (\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa}\right) = 0,\end{aligned} \tag{420}$$

which means that

$$\mathcal{H}_\alpha = -\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} \tag{421}$$

and

$$d\mathcal{H}_\alpha + d(\mathfrak{g}_\alpha\lrcorner\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} + (\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} = 0. \tag{422}$$

From Eq.(422) we have

$$\begin{aligned}d\mathcal{H}_\alpha &= -d(\mathfrak{g}_\alpha\lrcorner\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} - (\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} \\ &= -(\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} - d\left[(\mathfrak{g}_\alpha\lrcorner\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa}\right] + (\mathfrak{g}_\alpha\lrcorner\mathfrak{g}^\kappa)\wedge d\left(\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa}\right) \\ &= -(\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} - d\left[\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha}\right] + d\left(\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha}\right) \\ &= -(\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} + d\mathcal{H}_\alpha + d\left(\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha}\right),\end{aligned} \tag{423}$$

from where it follows the identity:

$$(\mathfrak{g}_\alpha\lrcorner d\mathfrak{g}^\kappa)\wedge\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\kappa} = d\left(\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha}\right). \tag{424}$$

We now return to Eq.(419) and recalling Eq.(408) we can write:

$$d(Z^{\alpha}\mathcal{H}_{\alpha} + dB) + \boldsymbol{\delta}\mathfrak{g}^{\alpha} \wedge \left(\frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^{\alpha}}\right) + \left(\frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{p}_{\alpha}}\right) \wedge \boldsymbol{\delta}\mathfrak{p}_{\alpha} = 0, \tag{425}$$

from where we get after some algebra

$$\boxed{\mathcal{H}_{\alpha}(\mathfrak{g}^{\alpha}, \mathfrak{p}_{\alpha}) = \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^{\alpha}} - (\mathfrak{g}_{\alpha}\lrcorner \wedge \mathfrak{p}_{\kappa}) \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{p}_{\kappa}},}$$
$$\boxed{B = Z^{\alpha}\mathfrak{p}_{\alpha}.} \tag{426}$$

7.2 The Quasi Local Energy

Now, let us investigate the meaning of the boundary term in Eq.(414). Consider an arbitrary spacelike hypersuface σ. Then we define

$$\begin{aligned}\mathbf{H} &= \int_{\sigma} (Z^{\alpha}\mathcal{H}_{\alpha} + dB) \\ &= \int_{\sigma} Z^{\alpha}\mathcal{H}_{\alpha} + \int_{\partial\sigma} B.\end{aligned} \tag{427}$$

If we take into account Eq.(421) we see that the first term in Eq.(421) is null when the field equations (for the free gravitational field) are satisfied and we are thus left with

$$\mathbf{E} = \int_{\partial\sigma} B, \tag{428}$$

which is called the quasi local energy. [59]

Before proceeding recall that if $\{\mathfrak{e}_{\alpha}\}$ is the dual basis of $\{\mathfrak{g}^{\alpha}\}$ we have

$$\mathfrak{g}^{\mathbf{0}}(\mathfrak{e}_{\mathbf{i}}) = 0, \ \mathbf{i} = 1, 2, 3. \tag{429}$$

Then, if we take $\mathbf{Z} = \mathfrak{e}_{\mathbf{0}}$ orthogonal to the hypersurface σ such that for each $p \in \sigma$, $T\sigma_p$ is generated by $\{\mathfrak{e}_{\mathbf{i}}\}$, we get recalling that in our theory it is $\mathfrak{p}_{\alpha} = \underset{g}{\star}\mathcal{S}_{\alpha}$ (Eq.(365)) that

$$\mathbf{E} = \int_{\partial\sigma} \underset{g}{\star}\mathcal{S}_0, \tag{430}$$

which we recognize as being the same conserved quantity as defined by P_0' in Eq.(371), which is the total energy contained in the gravitational plus matter fields for a solution of Einstein's equation such that the source term goes to zero at spatial infinity.

[59] For an up to date review on the concept of quasilocal energy in *GRT*, its success and drawbacks see [95].

7.3 Hamilton's Equations

Before we explore in details the meaning of this quasi local energy, let us remind that the role of an Hamiltonian density is to derive Hamilton's equation. We have

$$\boldsymbol{\delta}\ \mathbf{H} = \int_\sigma \boldsymbol{\delta}\mathcal{H} = \int_\sigma \boldsymbol{\delta}(Z^\alpha \mathcal{H}_\alpha + dB) \tag{431}$$

where $\boldsymbol{\delta}$ is an *arbitrary* variation not generated by the flow of the vector field $\mathbf{Z}$. To perform that variation we recall that we obtained above three different (but equivalent) expressions for $\mathcal{H}_\alpha$, namely Eq.(416), Eq.(418) and Eq.(428).

Taking into account that $\boldsymbol{\delta}\mathcal{L}_g(\mathfrak{g}^\alpha, d\mathfrak{g}^\alpha) = \boldsymbol{\delta}\mathfrak{L}_g(\mathfrak{g}^\alpha, \mathfrak{p}_\alpha)$, we have using Eq.(405) and Cartan's magical formula

$$\begin{aligned}\boldsymbol{\delta}\mathcal{H} &= \boldsymbol{\delta}\left(\pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha - Z\lrcorner\mathfrak{L}_g\right) \\ &= -Z\lrcorner\left(d(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) + \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha}\right) + \pounds_{\mathbf{Z}}\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha + \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha \\ &= -\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{p}_\alpha + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha - Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha}) + d[Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha)].\end{aligned} \tag{432}$$

This last equation can also be written using Eq.(408) as

$$\begin{aligned}\boldsymbol{\delta}\mathcal{H} =& -\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{p}_\alpha + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \\ & - Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{p}_\alpha}) + d[Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha)],\end{aligned} \tag{433}$$

and a simple calculation shows that this is also the form $\boldsymbol{\delta}\mathcal{H}$ that we get varying Eq.(422). From the above results it follows that

$$\begin{aligned}\boldsymbol{\delta}\mathbf{H} =& \int_\sigma (-\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{p}_\alpha + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha) \\ & \int_\sigma - Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{p}_\alpha}) \\ & + \int_{\partial\sigma} Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha).\end{aligned} \tag{434}$$

Thus, only if the field equations are satisfied (i.e., $\frac{\boldsymbol{\delta}\mathcal{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} = 0$ or $\frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} = 0$, $\frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{p}_\alpha} = 0$) and

$$\int_{\partial\sigma} Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) = 0, \tag{435}$$

for arbitrary variations $\boldsymbol{\delta}\mathfrak{g}^\alpha$ not necessarily vanishing at the boundary [74] $\partial\sigma$, it follows that the variation $\boldsymbol{\delta}\mathbf{H}$ gives Hamilton's equations:

$$\pounds_{\mathbf{Z}}\mathfrak{p}_\alpha = -\frac{\boldsymbol{\delta}\mathcal{H}}{\boldsymbol{\delta}\mathfrak{g}^\alpha},\ \ \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha = \frac{\boldsymbol{\delta}\mathcal{H}}{\boldsymbol{\delta}\mathfrak{p}_\alpha}. \tag{436}$$

Now the term $d[Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha)]$ comes as part of the variation of the term dB. Indeed, since

$$\begin{aligned} B &= Z^\alpha \mathfrak{p}_\alpha = (Z\lrcorner\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha \\ &= Z\lrcorner(\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) + \mathfrak{g}^\alpha \wedge (Z\lrcorner\mathfrak{p}_\alpha), \end{aligned} \tag{437}$$

it is

$$dB = d(Z\lrcorner(\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha)) + d(\mathfrak{g}^\alpha \wedge (Z\lrcorner\mathfrak{p}_\alpha)), \tag{438}$$

and

$$\begin{aligned} \boldsymbol{\delta} dB &= d(Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha)) + d(Z\lrcorner(\mathfrak{g}^\alpha \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha)) \\ &\quad + d(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge (Z\lrcorner\mathfrak{p}_\alpha)) + d(\mathfrak{g}^\alpha \wedge (Z\lrcorner\boldsymbol{\delta}\mathfrak{p}_\alpha)) \\ &= d(Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha)) + d((Z\lrcorner(\mathfrak{g}^\alpha) \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha) - \mathfrak{g}^\alpha \wedge (Z\lrcorner\boldsymbol{\delta}\mathfrak{p}_\alpha) + \mathfrak{g}^\alpha \wedge (Z\lrcorner\boldsymbol{\delta}\mathfrak{p}_\alpha)) \\ &\quad + d(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge (Z\lrcorner\mathfrak{p}_\alpha)) \\ &= d(Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha)) + d((Z\lrcorner(\mathfrak{g}^\alpha) \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha) + d(Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha)) \end{aligned} \tag{439}$$

This result is important since it shows that if Eq.(435) is not satisfied, we must modify in an appropriate way the boundary term B in Eq.(414) in order for that condition to hold. This was exactly the idea originally used by Nester and collaborators [9, 10, 60] in their proposal for the use of the quasi local energy concept. Here we analyze what happens in our theory when we choose, e.g., $Z = \mathfrak{g}^{\mathbf{0}}$. So, we examine the integrand in Eq.(435). We have:

$$Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) = (Z\lrcorner\boldsymbol{\delta}\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha - \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge (Z\lrcorner\mathfrak{p}_\alpha)\,. \tag{440}$$

and with $Z = \mathfrak{g}^{\mathbf{0}}$ it is

$$(Z\lrcorner\boldsymbol{\delta}\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha = \boldsymbol{\delta}[(Z\lrcorner\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha] - (\boldsymbol{\delta} Z\lrcorner\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha - (Z\lrcorner\mathfrak{g}^\alpha) \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha \tag{441}$$

So, if $Z = \mathfrak{g}^{\mathbf{0}}$.is maintained *fixed* on $\partial\sigma$ (for we used this hypothesis in deriving Eq.(439) we get

$$Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) = \boldsymbol{\delta}\mathfrak{p}_0 - \boldsymbol{\delta}\mathfrak{p}_0 = 0, \tag{442}$$

and thus the boundary term in $\boldsymbol{\delta}\mathbf{H}$ is *null*, thus implying the validity of Hamilton's equations of motion.

Remark 7.1 If we choose $Z \neq \mathfrak{g}^{\mathbf{0}}$ we will have in general that $Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \mathfrak{p}_\alpha) \neq 0$ and in this case a modification of the boundary term will indeed be necessary in order to get Hamilton's equations of motion. Following ideas first presented (for the best of our knowledge) in [52] concerning a simplectic framework for field theories, Chen and Nester presented the following prescription for modification of the $B(Z) = (Z\lrcorner\mathfrak{g}^\alpha) \wedge \mathfrak{p}_\alpha = Z^\alpha \wedge \mathfrak{p}_\alpha$ term: introduce a reference configuration $(\mathring{\mathfrak{g}}^\alpha, \mathring{\mathfrak{p}}_\alpha)$ and adjust $B(Z)$ to one of the two covariant simplectic forms,

$$B_{\mathfrak{g}^\alpha}(Z) = (Z\lrcorner\mathfrak{g}^\alpha) \wedge \triangle\mathfrak{p}_\alpha + \triangle\mathfrak{g}^\alpha \wedge (Z\lrcorner\mathring{\mathfrak{p}}_\alpha), \tag{443}$$

or

$$B_{\mathfrak{p}_\alpha}(Z) = (Z\lrcorner\mathring{\mathfrak{g}}^\alpha) \wedge \triangle\mathfrak{p}_\alpha + \triangle\mathfrak{g}^\alpha \wedge (Z\lrcorner\mathfrak{p}_\alpha), \tag{444}$$

where $\triangle\mathfrak{g}^\alpha = \mathfrak{g}^\alpha - \mathring{\mathfrak{g}}^\alpha$ and $\triangle\mathfrak{p}_\alpha = \mathfrak{p}_\alpha - \mathring{\mathfrak{p}}_\alpha$. Next introduce the improved Hamiltonian 3-forms:

$$\mathcal{H}_{\mathfrak{g}^\alpha}(Z) = Z^\mu\mathcal{H}_\mu + dB_{\mathfrak{g}^\alpha}(Z), \tag{445}$$

and

$$\mathcal{H}_{\mathfrak{p}_\alpha}(Z) = Z^\mu\mathcal{H}_\mu + dB_{\mathfrak{p}_\alpha}(Z). \tag{446}$$

For those improved Hamiltonians we immediately get for arbitrary variations (not generated by the flow of $\mathbf{Z}$)

$$\begin{aligned}\boldsymbol{\delta}\mathcal{H}_{\mathfrak{g}^\alpha}(Z) =& -\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{p}_\alpha + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \\ & - Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{p}_\alpha}) + d[Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \triangle\mathfrak{p}_\alpha)],\end{aligned} \tag{447}$$

and

$$\begin{aligned}\boldsymbol{\delta}\mathcal{H}_{\mathfrak{p}_\alpha}(Z) =& -\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{p}_\alpha + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \pounds_{\mathbf{Z}}\mathfrak{g}^\alpha \\ & - Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{g}^\alpha} + \boldsymbol{\delta}\mathfrak{p}_\alpha \wedge \frac{\boldsymbol{\delta}\mathfrak{L}_g}{\boldsymbol{\delta}\mathfrak{p}_\alpha}) - d[Z\lrcorner(\triangle\mathfrak{g}^\alpha \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha)].\end{aligned} \tag{448}$$

The variations $\boldsymbol{\delta}\mathcal{H}_{\mathfrak{g}^\alpha}(Z)$ and $\boldsymbol{\delta}\mathcal{H}_{\mathfrak{p}_\alpha}(Z)$ includes then, besides the field equations the following boundary terms,

$$dC_{\mathfrak{g}^\alpha} := d[Z\lrcorner(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \triangle\mathfrak{p}_\alpha)], \tag{449a}$$

$$-dC_{\mathfrak{p}_\alpha} := -d[Z\lrcorner(\triangle\mathfrak{g}^\alpha \wedge \boldsymbol{\delta}\mathfrak{p}_\alpha)] \tag{449b}$$

which are the projections on the boundary of covariant simplectic structures $dC_{\mathfrak{g}^\alpha}$ and $-dC_{\mathfrak{p}_\alpha}$, which simply reflect according the general methodology introduced in [52] the choice of the control mode.

Now, we can take arbitrary variations on the boundary such that $\boldsymbol{\delta}\mathfrak{g}^\alpha = o(1/r)$ and $\boldsymbol{\delta}\mathfrak{p}_\alpha = o(1/r^2)$ which give $\int_{\partial\sigma} B_{\mathfrak{g}^\alpha}(Z) = 0$ and $\int_{\partial\sigma} B_{\mathfrak{p}_\alpha}(Z) = 0$, warranting the validity of Hamilton's equations. Moreover, we have

$$B_{\mathfrak{g}^\alpha}(Z) - B_{\mathfrak{p}_\alpha}(Z) = Z\lrcorner(\triangle\mathfrak{g}^\alpha \wedge \triangle\mathfrak{p}_\alpha), \tag{450}$$

and $\triangle\mathfrak{g}^\alpha \wedge \triangle\mathfrak{p}_\alpha$ is null at spatial infinity if we $\mathring{\mathfrak{g}}^\alpha$ and $\mathring{\mathfrak{p}}_\alpha$ are the references concerning the Lorentz vacuum (Minkowski spacetime). Thus we can use $B_{\mathfrak{g}^\alpha}(Z)$ or $B_{\mathfrak{p}_\alpha}(Z)$ as good boundary terms, but each one gives a different concept of energy, which may be useful in applications.

7.4 The *ADM* Energy.

The relation of the previous sections with the original *ADM* formalism [1] can be seen as follows. Instead of choosing an orthonormal vector field $\mathbf{Z}$, start with a global timelike vector field $\mathbf{n} \in \sec TM$ such that $n = \boldsymbol{g}(\mathbf{N},) = N^2 dt \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$, with $N : \mathbb{R} \supset \mathbb{I} \to \mathbb{R}$, a positive function called the lapse function of M. Then $n \wedge dn = 0$ and according to Frobenius theorem n induces a foliation of M, i.e., topologically it is $M = \mathbb{I} \times \sigma_t$, where σ_t is a spacelike hypersurface with normal given by $\mathbf{n}$. Now, we can decompose any $A \in \sec \bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$ into a tangent component $\underline{A}$ to σ_t and an orthogonal component ${}^{\perp}A$ to σ_t by [44, 103]:

$$A = \underline{A} + {}^{\perp}A \; dt \wedge A_{\perp} \tag{451}$$

where

$$\underline{A} := n \lrcorner (dt \wedge A), \; {}^{\perp}A = dt \wedge A_{\perp} \tag{452}$$

$$A_{\perp} := n \lrcorner A. \tag{453}$$

Introduce also the parallel component $\underline{d}$ of the differential operator d by:

$$\underline{d}A := dt \wedge (n \lrcorner dA) \tag{454}$$

from where it follows (taking into account Cartan's magical formula) that

$$dA = dt \wedge (\pounds_{\mathbf{n}} \underline{A} - \underline{d} A_{\perp}) + \underline{dA}. \tag{455}$$

Call

$$\begin{aligned} \boldsymbol{m} &:= -\boldsymbol{g} + n \otimes n \\ &= \underline{\mathfrak{g}}^i \otimes \underline{\mathfrak{g}}_i, \end{aligned} \tag{456}$$

the first fundamental form on σ_t. and next introduce the Hodge dual operator associated[60] to $\boldsymbol{m}$ acting on the (horizontal forms) forms $\underline{A}$ by

$$\underset{\boldsymbol{m}}{\star} \underline{A} := \underset{g}{\star}\left(\frac{n}{N} \wedge \underline{A}\right). \tag{457}$$

At this point we come back to the Lagrangian density Eq.(412) and proceeding like in Sections 7.1 and 7.2, but now leaving $\boldsymbol{\delta} n^{\alpha}$ to be non null, we arrive to the following Hamiltonian density

$$\mathcal{H}(\underline{\mathfrak{g}}^i, \underline{\mathfrak{p}_i}) = \pounds_{\mathbf{n}} \underline{\mathfrak{g}}^i \wedge \underset{\boldsymbol{m}}{\star} \underline{\mathfrak{p}_i} - \mathcal{K}_g, \tag{458}$$

where

$$\underline{\mathfrak{g}}^i = dt \wedge (n \lrcorner \mathfrak{g}^i) = n^i dt, \tag{459}$$

[60] Recall that σ has an internal orientation compatible with the orientation of M.

and where K_g depends on $(n, \underline{d}n, \underline{\mathfrak{g}}^i, \underline{d\mathfrak{g}}^i, \pounds_{\mathbf{n}}\underline{\mathfrak{g}}^i)$. As in Section 7.1 we can show (after some tedious but straightforward algebra[61]) that $\mathcal{H}(\underline{\mathfrak{g}}^i, \underline{\mathfrak{p}_i})$ can be put into the form

$$\mathcal{H} = n^i \mathcal{H}_i + \underline{d}B', \tag{460}$$

with

$$\mathcal{H}_i = \frac{\boldsymbol{\delta} \mathcal{L}_g}{\boldsymbol{\delta} \mathfrak{g}^i} = \frac{\boldsymbol{\delta} \mathcal{K}_g}{\boldsymbol{\delta} n^i} \tag{461}$$

and

$$B' = -N \underline{\mathfrak{g}_i} \wedge \underset{m}{\underline{\star d\mathfrak{g}^i}} \tag{462}$$

Then, on shell, i.e., when the field equations are satisfied we get

$$\mathbf{E}' = -\int_{\partial\sigma_t} N \underline{\mathfrak{g}_i} \wedge \underset{m}{\underline{\star d\mathfrak{g}^i}}, \tag{463}$$

which is exactly the ADM energy, as can be seem if we take into account that taking $\partial\sigma_t$ as an two-sphere at the infinity we have (using coordinates in the ELP gauge) $\underline{\mathfrak{g}_i} = h_{ij}\underline{d}\mathrm{x}^j$ and $h_{ij}, N \to 1$. Then

$$\underline{\mathfrak{g}_i} \wedge \underset{m}{\underline{\star d\mathfrak{g}^i}} = h^{ij}\left(\frac{\partial h_{ij}}{\partial \mathrm{x}^k} - \frac{\partial h_{ik}}{\partial \mathrm{x}^j}\right) \underset{m}{\star} \underline{\mathfrak{g}^k} \tag{464}$$

and under the above conditions we have the ADM formula

$$\mathbf{E}' = \int_{\partial\sigma_t} \left(\frac{\partial h_{ik}}{\partial \mathrm{x}^i} - \frac{\partial h_{ii}}{\partial \mathrm{x}^k}\right) \underset{m}{\star} \underline{\mathfrak{g}^k}., \tag{465}$$

which as well known[62] is positive definite [107]. When we choose $n = \mathfrak{g}^0$ as in the previous section we see that $\mathbf{E}' = \mathbf{E}$.

8 Conclusions

In this paper we present a theory of the gravitational field where this field is mathematically represented a $(1,1)$-extensor field $\boldsymbol{h}$ living in Minkowski spacetime $(M, \boldsymbol{\eta}, D, \tau_\eta, \uparrow)$ and describing a plastic distortion of a medium that we called the Lorentz vacuum. We showed (Remark 6.2) that the presence of a nontrivial distortion field can be described by distinct effective geometrical structures $(M = \mathbb{R}^4, \boldsymbol{g} = \boldsymbol{h}\eta\boldsymbol{h}^\dagger, \nabla, \tau_g)$ which may associated to a Lorentzian spacetime, or a teleparallel spacetime or even a more general Riemann-Cartan-Weyl

[61] Details may be found in [103].

[62] As shown by Wallner ([103]) the formalism used in this section permits to give an alternative (and very simple) proof of the positive mass theorem, different from the one in [107].

spacetime [63]. We postulated a Lagrangian density for the distortion field in interaction with matter, and developed with details the mathematical theory (the multiform and extensor calculus) necessary to obtain the equations of motion of the theory from a variational principle. Moreover, we showed that when our theory is written in terms of the global potentials $\mathfrak{g}^\alpha = \boldsymbol{h}(\boldsymbol{\vartheta}^\alpha) \in \sec \bigwedge^1 T^*M$ obtained by deformation of the (continuum) cosmic lattice describing the Lorentz vacuum in its ground state the Lagrangian density contains a term of the Yang-Mills type plus a gauge fixing term and plus a term describing the interaction of the 'vorticities' of the potentials $\mathfrak{g}^\alpha$ and no connection associated to any geometrical structure appears in it. We proved that in our theory (in contrast with GRT) there are trustworthy conservation laws of energy-momentum and angular momentum. We showed moreover how the results obtained for the energy-momentum conservation law from the Lagrangian formalism appears directly in the Hamiltonian formalism of the theory and how the energy obtained through the Hamiltonian formalism is also related to the concept of ADM energy used in GRT. We also discussed in Appendix F why a gauge theory of gravitation described in [21, 48] and apparently similar to ours do not work. To end we would like to observe that eventually some readers will not fell happy for seeing us to introduce an universal medium filling all spacetime, because it eventually looks like the revival of ether of the XIX^{th} century. To those people we ask to read urgently [24] and also to study the arguments for the existence of such a medium (that quantum physicists call quantum vacuum) in, e.g., [4, 18, 56, 57, 87, 88, 89, 102] and also analyze references [100, 101] which show that if we interpret particles as defects in the medium[64], then some characteristic relativistic and quantum field theory properties of those objects emerge in a natural way. Moreover, also there is some hope to possibly describe the electromagnetic field as some kind of distortion of that medium. We will come back to those issues in a forthcoming paper. We hope to have motivated the reader to seriously consider the idea that eventually is time to deconstruct the idea that gravitation must be necessarily associated with the geometry of spacetime and look for its real nature.

Acknowledgement The authors are grateful to Dr. E. A. Notte-Cuello, Dr. A. M. Moya, Dr. R. da Rocha, Dr. Q. A. G. Souza and Professor J. Vaz Jr. for many useful discussions on the subject of this paper.

[63] In that latter case we can get equations of motion similar to Einstein equations but where the matter tensor appears as a combination of terms proportional to the torsion tensor, that as well known in continuum theories describe some special defects in the medium. This possibility will be discussed with more details in another publication. For a simple model where the gravitational field is represented by the nonmetricity tensor of a connection see Appendix G.

[64] Those topological defects introduce obstructions for the Pfaff topology generated by the $\mathfrak{g}^\alpha$ [50, 51], a subject that we shall discuss in another publication.

A May a Torus with Null Riemann Curvature Live on $\mathbb{E}^3$?

We can also give a flat Riemann curvature tensor for[65] $\mathcal{T}_3 \simeq S^1 \times S^1 \simeq \mathcal{T} \simeq \mathcal{T}_2$. First parametrize $\mathcal{T}_3$ by (x^1, x^2), $x^1, x^2 \in \mathbb{R}$, such that its coordinates in $\mathbb{R}^3$ are $\mathbf{x}(x^1, x^2) = \mathbf{x}(x^1 + 2\pi, x^2 + 2\pi)$ and

$$\mathbf{x}(x^1, x^2) = (h(x^1) \cos x^2, h(x^1) \sin x^2, l(x^1), \tag{466}$$

with

$$h(x^1) = R + r \cos x^1,\ l(x^1) = r \sin x^1, \tag{467}$$

where *here* R and r are real positive constants and $R > r$ (See Figure 1).

Figure 1: Some Possible *GSS* for the Torus $\mathcal{T}_3$ living in $\mathbb{E}^3$ where the grid defines the parallelism

Now, $T\mathcal{T}_3$ (the tangent bundle of $\mathcal{T}_3$) has a global coordinate basis $\{\partial/\partial x^1, \partial/\partial x^2\}$ and the induced metric on $\mathcal{T}_3$ is easily found as

$$\boldsymbol{g} = r^2 dx^1 \otimes dx^1 + (R + r \cos x^1)^2 dx^2 \otimes dx^2 \tag{468}$$

Now, let us introduced a connection ∇ on $\mathcal{T}_3$ such that

$$\nabla_{\partial/\partial x^i} \partial/\partial x^j = 0. \tag{469}$$

With respect to this connection it is immediate to verify that its torsion and curvature tensors are null, but the *nonmetricity* of the connection is non null. Indeed,

$$\nabla_{\partial/\partial x^i} \boldsymbol{g} = -2(R + r \cos x^1) \sin x^1. \tag{470}$$

[65] Recall that $\mathcal{T}_1$ and $\mathcal{T}_2$ have been defined in Section 1.1.

So, in this case we have a particular Riemann-Cartan-Weyl *GSS* $(\mathcal{T}_3, \mathbf{g}, \nabla)$ with $\mathfrak{A} \neq 0$, $\Theta = 0$, $\mathfrak{R} = 0$.

We can alternatively define the parallelism rule on $\mathcal{T}_3$ by introducing this time a metric compatible connection ∇' as follows. First we introduce the global orthonormal basis $\{\mathbf{e}_i\}$ for $T\mathcal{T}_3$ with

$$\mathbf{e_1} = \frac{1}{r}\frac{\partial}{\partial x^1},\ \mathbf{e_2} = \frac{1}{(R + r\cos x^1)}\frac{\partial}{\partial x^2}, \tag{471}$$

and then postulate that

$$\nabla'_{\mathbf{e}_i}\mathbf{e_j} = 0. \tag{472}$$

Since

$$[\mathbf{e_1}, \mathbf{e_2}] = \frac{r\sin x^1}{r(R + r\cos x^1)}\mathbf{e_2} \tag{473}$$

we immediately get that the torsion operator of ∇' is

$$\overset{\nabla'}{\tau}(\mathbf{e_1}, \mathbf{e_2}) = [\mathbf{e_1}, \mathbf{e_2}] = \frac{R\sin x^1}{r(R + r\cos x^1)}\mathbf{e_2}, \tag{474}$$

and the non null components of the torsion tensor are $T^{\mathbf{2}}_{\mathbf{12}}$ and $T^{\mathbf{2}}_{\mathbf{21}} = -T^{\mathbf{2}}_{\mathbf{12}}$,

$$T^{\mathbf{2}}_{\mathbf{12}} = \overset{\nabla'}{\Theta}(\theta^{\mathbf{2}}, \mathbf{e_1}, \mathbf{e_2}) = -\theta^{\mathbf{2}}(\tau(\mathbf{e_1}, \mathbf{e_2})) = -\theta^{\mathbf{2}}\left(-[\mathbf{e_1}, \mathbf{e_2}]\right) = -\frac{R\sin x^1}{r(R + r\cos x^1)}. \tag{475}$$

Also it is trivial to see that the curvature operator is null and that $\nabla'\mathbf{g} = 0$ and in his case we have a particular Riemann-Cartan *MCGSS* $(\mathcal{T}_3, \mathbf{g}, \nabla')$, with $\Theta \neq 0$ and with a null curvature tensor.

B The Levi-Civita and the Nunes Connection on the Punctured Sphere

First consider S^2, an sphere of radius $R = 1$ embedded in $\mathbb{R}^3$. Let $(x^1, x^2) = (\vartheta, \varphi)$ $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$, be the standard spherical coordinates of S^2, which covers all the open set U which is S^2 with the *exclusion* of a semi-circle uniting the *north* and south *poles*.

Introduce the *coordinate bases*

$$\{\boldsymbol{\partial}_\mu\}, \{\theta^\mu = dx^\mu\} \tag{476}$$

for T^*U and T^*U. Next introduce the *orthonormal bases* $\{\mathbf{e_a}\}, \{\theta^{\mathbf{a}}\}$ for T^*U and T^*U with

$$\mathbf{e_1} = \boldsymbol{\partial}_1,\ \mathbf{e_2} = \frac{1}{\sin x^1}\boldsymbol{\partial}_2, \tag{477a}$$

$$\theta^{\mathbf{1}} = dx^1,\ \theta^{\mathbf{2}} = \sin x^1 dx^2. \tag{477b}$$

Then,

$$
\begin{aligned}
[\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}] &= c_{\mathbf{ij}}^{\mathbf{k}} \boldsymbol{e}_{\mathbf{k}}, \\
c_{\mathbf{12}}^{\mathbf{2}} &= -c_{\mathbf{21}}^{\mathbf{2}} = -\cot x^{\mathbf{1}}.
\end{aligned} \tag{478}
$$

Moreover the metric $\boldsymbol{g} \in \sec T_2^0 S^2$ inherited from the ambient Euclidean metric is:

$$
\begin{aligned}
\boldsymbol{g} &= dx^1 \otimes dx^1 + \sin^2 x^1 dx^2 \otimes dx^2 \\
&= \theta^{\mathbf{1}} \otimes \theta^{\mathbf{1}} + \theta^{\mathbf{2}} \otimes \theta^{\mathbf{2}}.
\end{aligned} \tag{479}
$$

The Levi-Civita connection D of $\boldsymbol{g}$ has the following non null connections coefficients $\Gamma_{\mu\nu}^{\rho}$ in the coordinate basis (just introduced):

$$
\begin{aligned}
& D_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu = \Gamma_{\mu\nu}^{\rho} \boldsymbol{\partial}_\rho, \\
& \quad \Gamma_{21}^{2} = \Gamma_{\theta\varphi}^{\varphi} = \Gamma_{12}^{2} = \Gamma_{\varphi\theta}^{\varphi} = \cot\vartheta,\ \Gamma_{22}^{1} = \Gamma_{\varphi\varphi}^{\vartheta} = -\cos\vartheta \sin\vartheta.
\end{aligned} \tag{480}
$$

Also, in the basis $\{\boldsymbol{e}_{\mathbf{a}}\}$, $D_{\boldsymbol{e}_{\mathbf{i}}} \boldsymbol{e}_{\mathbf{j}} = \omega_{\mathbf{ij}}^{\mathbf{k}} \boldsymbol{e}_{\mathbf{k}}$ and the non null coefficients are:

$$
\omega_{\mathbf{21}}^{\mathbf{2}} = \cot\vartheta,\ \omega_{\mathbf{22}}^{\mathbf{1}} = -\cot\vartheta. \tag{481}
$$

The torsion and the (Riemann) curvature tensors of D are

$$
\Theta(\theta^{\mathbf{k}}, \boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) = \theta^{\mathbf{k}}(\tau(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}})) = \theta^{\mathbf{k}} \left(D_{\boldsymbol{e}_{\mathbf{j}}} \boldsymbol{e}_{\mathbf{i}} - D_{\boldsymbol{e}_{\mathbf{i}}} \boldsymbol{e}_{\mathbf{j}} - [\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}] \right), \tag{482}
$$

$$
\mathbf{R}(\boldsymbol{e}_{\mathbf{k}}, \theta^{\mathbf{a}}, \boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) = \theta^{\mathbf{a}} \left(\left[D_{\boldsymbol{e}_{\mathbf{i}}} D_{\boldsymbol{e}_{\mathbf{j}}} - D_{\boldsymbol{e}_{\mathbf{j}}} D_{\boldsymbol{e}_{\mathbf{i}}} - D_{[\boldsymbol{e}_{\mathbf{i}},\ \boldsymbol{e}_{\mathbf{j}}]} \right] \boldsymbol{e}_{\mathbf{k}} \right), \tag{483}
$$

which results in $\mathcal{T} = 0$ and that the non null components of $\mathbf{R}$ are $R_{\mathbf{1}\ \mathbf{21}}^{\ \mathbf{1}} = -R_{\mathbf{1}\ \mathbf{12}}^{\ \mathbf{1}} = R_{\mathbf{1}\ \mathbf{12}}^{\ \mathbf{2}} = -R_{\mathbf{1}\ \mathbf{12}}^{\ \mathbf{2}} = -1$.

Since the Riemann curvature tensor is non null the parallel transport of a given vector depends on the path to be followed. We say that a vector (say $\mathbf{v}_0$) is parallel transported along a generic path $\mathbb{R} \supset I \mapsto \gamma(s) \in \mathbb{R}^3$ (say, from $A = \gamma(0)$ to $B = \gamma(1)$) with tangent vector $\gamma_*(s)$ (at $\gamma(s)$) if it determines a vector field $\mathbf{V}$ along γ satisfying

$$
D_{\gamma_*} \mathbf{V} = 0, \tag{484}
$$

and such that $\mathbf{V}(\gamma(0)) = \mathbf{v}_0$. When the path is a geodesic[66] of the connection D, i.e., a curve $\mathbb{R} \supset I \mapsto c(s) \in \mathbb{R}^3$ with tangent vector $c_*(s)$ (at $c(s)$) satisfying

$$
D_{c_*} c_* = 0, \tag{485}
$$

the parallel transported vector along a c forms a constant angle with c_*. Indeed, from Eq.(484) it is $\gamma_* \underset{\boldsymbol{g}}{\cdot} D_{\gamma_*} \mathbf{V} = 0$. Then taking into account Eq.(485) it follows that

$$
D_{\gamma_*}(\gamma_* \underset{\boldsymbol{g}}{\cdot} \mathbf{V}) = 0.
$$

Figure 2: Levi-Civita and Nunes transport of a vector $\mathbf{v}_0$ starting at p through the paths *psr* and *pqr*. Levi-Civita transport through psr leads to $\mathbf{v}_1$ whereas Nunes transport leads to $\mathbf{v}_2$. Along *pqr* both Levi-Civita and Nunes transport agree and leads to $\mathbf{v}_2$.

i.e., $\gamma_* \underset{g}{\cdot} \mathbf{V} =$ *constant*.This is clearly illustrated in Figure 1 (adapted from [3]).

Consider next the manifold $\mathring{S}^2 = \{S^2 \backslash \text{north pole} + \text{south pole}\} \subset \mathbb{R}^3$, which is an sphere of unitary radius $R = 1$, but this time *excluding* the north and south poles. Let again $\boldsymbol{g} \in \sec T_2^0 \mathring{S}^2$ be the metric field on $\mathring{S}^2$ inherited from the ambient space $\mathbb{R}^3$ and introduce on $\mathring{S}^2$ the Nunes (or navigator) connection ∇ defined by the following parallel transport rule: a vector at an arbitrary point of $\mathring{S}^2$ is parallel transported along a curve γ, if it determines a vector field on γ such that at any point of γ the angle between the transported vector and the vector tangent to the latitude line passing through that point is constant during the transport. This is clearly illustrated in Figure 3. and to distinguish the Nunes transport from the Levi-Civita transport we ask also for the reader to study with attention the caption of Figure 1.

We recall that from the calculation of the Riemann tensor $\mathbf{R}$ it follows that the structures $(\mathring{S}^2, \boldsymbol{g}, D, \tau_{\boldsymbol{g}})$ and also $(S^2, \boldsymbol{g}, D, \tau_{\boldsymbol{g}})$ are Riemann spaces of *constant curvature*. We now show that the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}})$ is a *teleparallel space*[67], with zero Riemann curvature tensor, but non zero torsion tensor.

Indeed, from Figure 2 it is clear that (a) if a vector is transported along the *infinitesimal* quadrilateral *pqrs* composed of latitudes and longitudes, first starting from p along *pqr* and then starting from p along *psr* the parallel trans-

[66]We recall that a geodesic of D also determines the minimal distance (as given by the metric $\mathbf{g}$) between any two points on S^2.

[67]As recalled in Section 1, a teleparallel manifold M is characterized by the existence of global vector fields which is a basis for T_xM for any $x \in M$. The reason for considering $\mathring{S}^2$ for introducing the Nunes connection is that as well known (see, e.g., [19]) S^2 does not admit a continuous vector field that is nonnull at on points of it.

Figure 3: Characterization of the Nunes connection.

ported vectors that result in both cases will coincide (study also the caption of Figure 2.

Now, the vector fields $\boldsymbol{e}_{\mathbf{1}}$ and $\boldsymbol{e}_{\mathbf{2}}$ in Eq.(477a) define a basis for each point p of $T_p\mathring{S}^2$ and ∇ is clearly characterized by:

$$\nabla_{\boldsymbol{e}_{\mathbf{j}}}\boldsymbol{e}_{\mathbf{i}} = 0. \tag{486}$$

The components of curvature operator are:

$$\overset{\nabla}{\mathbf{R}}(\boldsymbol{e}_{\mathbf{k}}, \theta^{\mathbf{a}}, \boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) = \theta^{\mathbf{a}}\left(\left[\nabla_{\boldsymbol{e}_{\mathbf{i}}}\nabla_{\boldsymbol{e}_{\mathbf{j}}} - \nabla_{\boldsymbol{e}_{\mathbf{j}}}\nabla_{\boldsymbol{e}_{\mathbf{i}}} - \nabla_{[\boldsymbol{e}_{\mathbf{i}},\boldsymbol{e}_{\mathbf{j}}]}\right]\boldsymbol{e}_{\mathbf{k}}\right) = 0, \tag{487}$$

and the torsion operator is

$$\begin{aligned} \overset{\nabla}{\tau}(\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}) &= \nabla_{\boldsymbol{e}_{\mathbf{j}}}\boldsymbol{e}_{\mathbf{i}} - \nabla_{\boldsymbol{e}_{\mathbf{i}}}\boldsymbol{e}_{\mathbf{j}} - [\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}] \\ &= [\boldsymbol{e}_{\mathbf{i}}, \boldsymbol{e}_{\mathbf{j}}], \end{aligned} \tag{488}$$

which gives for the components of the torsion tensor, $T_{\mathbf{12}}^{\mathbf{2}} = -T_{\mathbf{12}}^{\mathbf{2}} = \cot\vartheta$. It follows that $\mathring{S}^2$ considered as part of the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla, \tau_g)$ is *flat* (but has torsion)!

If you still need more details to grasp this last result, consider Figure 2 (b) which shows the standard parametrization of the points p, q, r, s in terms of the spherical coordinates introduced above. According to the geometrical meaning of torsion, its value at a given point is determined by calculating the

difference between the (infinitesimal)[68] vectors $pr_1$and pr_2. If the vector pq is transported along ps one get (recalling that $R = 1$) the vector $\mathbf{v} = sr_1$ such that $|\boldsymbol{g}(\mathbf{v},\mathbf{v})|^{\frac{1}{2}} = \sin\vartheta\triangle\varphi$. On the other hand, if the vector ps is transported along pq one get the vector $qr_2 = qr$. Let $\mathbf{w} = sr$. Then,

$$|\boldsymbol{g}(\mathbf{w},\mathbf{w})| = \sin(\vartheta - \triangle\vartheta)\triangle\varphi \simeq \sin\vartheta\triangle\varphi - \cos\vartheta\triangle\vartheta\triangle\varphi, \tag{489}$$

Also,

$$\mathbf{u} = r_1 r_2 = -u(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}),\ u = |\boldsymbol{g}(\mathbf{u},\mathbf{u})| = \cos\vartheta\triangle\vartheta\triangle\varphi. \tag{490}$$

Then, the connection ∇ of the structure $(\mathring{S}^2, \boldsymbol{g}, \nabla, \tau_{\boldsymbol{g}})$ has a non null torsion tensor $\bar{\mathcal{T}}$. Indeed, the component of $\mathbf{u} = r_1 r_2$ in the direction $\partial/\partial\varphi$ is precisely $\bar{T}^{\varphi}_{\vartheta\varphi}\triangle\vartheta\triangle\varphi$. So, one get (recalling that $\nabla_{\partial j}\partial_i = \Gamma^k_{ji}\partial_k$)

$$\bar{T}^{\varphi}_{\vartheta\varphi} = \left(\Gamma^{\varphi}_{\vartheta\varphi} - \Gamma^{\varphi}_{\varphi\vartheta}\right) = -\cot\theta. \tag{491}$$

To end this Appendix it is worth to show that ∇ is metrical compatible, i.e., that $\nabla\boldsymbol{g} = 0$. Indeed, we have:

$$\begin{aligned} 0 = &\ \nabla_{\boldsymbol{e}_\mathbf{c}}\boldsymbol{g}(\boldsymbol{e}_\mathbf{i},\boldsymbol{e}_\mathbf{j}) = (\nabla_{\boldsymbol{e}_\mathbf{c}}\boldsymbol{g})(\boldsymbol{e}_\mathbf{i},\boldsymbol{e}_\mathbf{j}) + \ \boldsymbol{g}(\nabla_{\boldsymbol{e}_\mathbf{c}}\boldsymbol{e}_\mathbf{i},\boldsymbol{e}_\mathbf{j}) + \ \boldsymbol{g}(\boldsymbol{e}_\mathbf{i},\nabla_{\boldsymbol{e}_\mathbf{c}}\boldsymbol{e}_\mathbf{j}) \\ &= (\nabla_{\boldsymbol{e}_\mathbf{c}}\boldsymbol{g})(\boldsymbol{e}_\mathbf{i},\boldsymbol{e}_\mathbf{j}). \end{aligned} \tag{492}$$

C Gravitational Theory for Independent $\boldsymbol{h}$ and Ω Fields on Minkowski Spacetime

In this section we present for completeness a theory where the dynamics of the gravitational field is such that it must be described by two independent fields, namely the plastic distortion field $\boldsymbol{h}$ (more precisely $\boldsymbol{h}^{\clubsuit}$) and a $\varkappa\eta$-gauge rotation field $\Omega \equiv \overset{\varkappa\eta}{\Omega}$ defined on the canonical space U. Those two independent fields distorts the Lorentz vacuum creating an effective Minkowski-Cartan *MCGSS* $(U, \eta, \varkappa)$ or equivalently a Lorentz-Cartan *MCGSS* $(U, \boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h}, \gamma)$ which is viewed as gauge equivalent to $(U, \eta, \varkappa)$ as already discussed in the main text. From the mathematical point of view our theory is then described by a Lagrangian where the fields $\boldsymbol{h}$ and Ω are the dynamic variables to be varied in the action functional. The Lagrangian is postulated to be the scalar functional

$$[\boldsymbol{h}^{\clubsuit}, \Omega, \cdot\partial\Omega] \mapsto \mathfrak{L}[\boldsymbol{h}^{\clubsuit}, \Omega, \cdot\partial\Omega] = \mathcal{R}\det[\boldsymbol{h}], \tag{493}$$

[68]This wording, of course, means that those vectors are identified as elements of the appropriate tangent spaces.

where $\mathcal{R}(=\overset{\gamma_h}{\mathcal{R}})$ (given by Eq.(293)) is expressed as a functional of the fields $\boldsymbol{h}^{\clubsuit}$, Ω and $\cdot\partial\Omega$, i.e.,

$$\begin{aligned}\mathcal{R} &= \underline{\boldsymbol{h}}^{\clubsuit}(\partial_a\wedge\partial_b)\cdot\mathcal{R}_2(a\wedge b)\\ &= \underline{\boldsymbol{h}}^{\clubsuit}(\partial_a\wedge\partial_b)\cdot(a\cdot\partial\Omega(b)-b\cdot\partial\Omega(a)-\Omega([a,b])+\Omega(a)\underset{\eta}{\times}\Omega(b)),\end{aligned}\tag{494}$$

and $\det[\boldsymbol{h}]$, as in Section 5.1 is expressed as a functional of $\boldsymbol{h}^{\clubsuit}$, i.e., $\det[\boldsymbol{h}] = (\det[\boldsymbol{h}^{\clubsuit}])^{-1}$.

The Lagrangian formalism for extensor fields supposes the existence of an action functional

$$\mathcal{A}=\int_U \mathfrak{L}[\boldsymbol{h}^{\clubsuit},\Omega,\cdot\partial\Omega]\;\tau=\int_U \mathcal{R}\det[\boldsymbol{h}]\;\tau \tag{495}$$

which according to the principle of stationary action gives for any one of the dynamic variables a contour problem.

For the dynamic variable $\boldsymbol{h}^{\clubsuit}$ it is

$$\int_U \boldsymbol{\delta}^w_{h^{\clubsuit}}\mathfrak{L}[\boldsymbol{h}^{\clubsuit},\Omega,\cdot\partial\Omega]\;\tau=0, \tag{496}$$

for any smooth $(1,1)$-extensor field w such that $w|_{\partial U}=0$.

For the dynamic variable Ω it is

$$\int_U \boldsymbol{\delta}^B_{\Omega}\mathfrak{L}[\boldsymbol{h}^{\clubsuit},\Omega,\cdot\partial\Omega]\;\tau=0, \tag{497}$$

for any smooth $(1,2)$-extensor field B, such that $B|_{\partial U}=0$.

We now show that Eqs.(496) and (497) give to coupled differential equations for the fields $\boldsymbol{h}^{\clubsuit}$ and Ω.

Of course, the solution of the contour problems given by Eqs.(496) and (497) is obtained following the traditional path, i.e., by using the appropriated variational formulas, the Gauss-Stokes and a fundamental lemma of integration theory (already mentioned in Section 5.1).

First we calculate the variation in Eq.(496). We have:

$$\begin{aligned}\boldsymbol{\delta}^w_{h^{\clubsuit}}(\underline{\boldsymbol{h}}^{\clubsuit}(\partial_a\wedge\partial_b)\cdot\mathcal{R}_2(a\wedge b)\det[\boldsymbol{h}]) &= (\boldsymbol{\delta}^w_{h^{\clubsuit}}\underline{\boldsymbol{h}}^{\clubsuit}(\partial_a\wedge\partial_b))\cdot\mathcal{R}_2(a\wedge b)\det[\boldsymbol{h}]\\ &\quad+\underline{\boldsymbol{h}}^{\clubsuit}(\partial_a\wedge\partial_b)\cdot\mathcal{R}_2(a\wedge b)(\delta^w_{\boldsymbol{h}^{\clubsuit}}\det[\boldsymbol{h}]).\end{aligned}\tag{498}$$

For the scalar product in the first member of Eq.(498), an algebraic manipulation analogous to the one already utilized for deriving Eq.(320) gives

$$(\boldsymbol{\delta}^w_{h^{\clubsuit}}\underline{\boldsymbol{h}}^{\clubsuit}(\partial_a\wedge\partial_b))\cdot\mathcal{R}_2(a\wedge b)=2w(\partial_a)\cdot\mathcal{R}_1(a), \tag{499}$$

where $\mathcal{R}(a)$ is the Ricci gauge field (in the Lorentz-Cartan *MCGSS*).

The calculation of $\boldsymbol{\delta}^{w}_{h^\clubsuit} \det[\boldsymbol{h}]$ has already been obtained (recall Eq.(327)) . So, substituting Eqs.(499) and (327) in Eq.(498) and recalling the definition of $\mathcal{G} \equiv \overset{\gamma_h}{\mathcal{G}}$ (Eq.(294)) we get that

$$\begin{aligned}
\boldsymbol{\delta}^{w}_{h^\clubsuit}(\mathcal{R}\det[\boldsymbol{h}]) &= 2w(\partial_a)\cdot\mathcal{R}_1(a)\det[\boldsymbol{h}] - \mathcal{R}w(\partial_a)\cdot\boldsymbol{h}(a)\det[\boldsymbol{h}] \\
&= 2w(\partial_a)\cdot(\mathcal{R}_1(a) - \frac{1}{2}\boldsymbol{h}(a)\mathcal{R})\det[\boldsymbol{h}] \\
\boldsymbol{\delta}^{w}_{h^\clubsuit}(\mathcal{R}\det[\boldsymbol{h}]) &= 2w(\partial_a)\cdot\mathcal{G}(a)\det[\boldsymbol{h}]. \qquad (500)
\end{aligned}$$

Substituting Eq.(500) in Eq.(496), the contour problem for the dynamic variable $\boldsymbol{h}^\clubsuit$ becomes

$$\int_{\mathcal{U}} w(\partial_a)\cdot\mathcal{G}(a)\det[\boldsymbol{h}]\;\tau = 0, \qquad (501)$$

for any w, and since w is arbitrary, recalling a fundamental lemma of integration theory, we get

$$\mathcal{G}(a) = 0. \qquad (502)$$

This field equation looks like the one already obtained for $\boldsymbol{h}$ in Section 6.1., but here it is not a differential equation for the field $\boldsymbol{h}$. To continue recall that Eq.(502) is equivalent to $\mathcal{R}(a) = 0$.

Indeed, taking the scalar $\boldsymbol{h}^\clubsuit$-divergent of $\mathcal{R}_1(a)$ we have

$$\begin{aligned}
\mathcal{G}(a) = 0 &\Rightarrow \mathcal{R}_1(a) - \frac{1}{2}\boldsymbol{h}(a)\mathcal{R} = 0 \\
&\Rightarrow \boldsymbol{h}^\clubsuit(\partial_a)\cdot\mathcal{R}_1(a) - \frac{1}{2}\boldsymbol{h}^\clubsuit(\partial_a)\cdot\boldsymbol{h}(a)\mathcal{R} = 0 \\
&\Rightarrow \mathcal{R} - \frac{1}{2}4\mathcal{R} = 0 \\
&\Rightarrow \mathcal{R} = 0,
\end{aligned}$$

which implies

$$\mathcal{R}_1(a) = 0. \qquad (503)$$

Also it quite clear that Eq.(503) implies Eq.(502).

Let us express now Eq.(503) in terms of the *canonical* 1-forms of $\boldsymbol{h}$ (i.e., $\boldsymbol{h}(\boldsymbol{\vartheta}^0),\ldots,\boldsymbol{h}(\boldsymbol{\vartheta}^3)$) and the canonical *biforms* of Ω (i.e., $\Omega(\vartheta_0),\ldots,\Omega(\vartheta_3)$),

$$\begin{aligned}
\mathcal{R}_1(a) = 0 &\Leftrightarrow \mathcal{R}_1(\vartheta_\nu) = 0 \\
&\Leftrightarrow \boldsymbol{h}^\clubsuit(\boldsymbol{\vartheta}^\mu)\lrcorner\mathcal{R}_2(\vartheta_\mu\wedge\vartheta_\nu) = 0,
\end{aligned}$$

i.e.,

$$\boldsymbol{h}^\clubsuit(\boldsymbol{\vartheta}^\mu)\lrcorner(\vartheta_\mu\cdot\partial\Omega(\vartheta_\nu) - \vartheta_\nu\cdot\partial\Omega(\vartheta_\mu) + \Omega(\vartheta_\mu)\underset{\eta}{\times}\Omega(\vartheta_\nu)) = 0. \qquad (504)$$

Eq.(504) resumes a set of four coupled multiform equations which are algebraic on the canonical *1*-forms of $\boldsymbol{h}$ and differential on the canonical biforms of Ω.

Let us now calculate the variation in Eq.(497). We have

$$\boldsymbol{\delta}_{\Omega}^{B}(\underline{\boldsymbol{h}}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot \mathcal{R}_2(a \wedge b) \det[\boldsymbol{h}]) = \underline{\boldsymbol{h}}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot (\boldsymbol{\delta}_{\Omega}^{B} \mathcal{R}_2(a \wedge b)) \det[\boldsymbol{h}]. \quad (505)$$

For the scalar product on the right side of Eq.(505), utilizing the commutation property involving $\boldsymbol{\delta}_{\Omega}^{B}$ and $a\cdot\partial$, the property $\boldsymbol{\delta}_{\Omega}^{B}\Omega(a) = B(a)$ and the definition of the covariant derivative operator $\mathcal{D}_a \equiv \overset{\varkappa\eta}{\mathcal{D}}_a$, i.e., $\mathcal{D}_a X = a\cdot\partial X + \Omega(a) \underset{\eta}{\times} X$, we have

$$\begin{aligned}
\underline{\boldsymbol{h}}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot (\boldsymbol{\delta}_{\Omega}^{B} \mathcal{R}_2(a \wedge b)) &= \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot (\boldsymbol{\delta}_{\Omega}^{B} \mathcal{R}_2(\vartheta_{\mu} \wedge \vartheta_{\nu})) \\
&= \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot (\vartheta_{\mu} \cdot \partial B(\vartheta_{\nu}) - \vartheta_{\nu} \cdot \partial B(\vartheta_{\mu}) + \\
&\quad B(\vartheta_{\mu}) \underset{\eta^{-1}}{\times} \Omega(\vartheta_{\nu}) + \Omega(\vartheta_{\mu}) \underset{\eta^{-1}}{\times} B(\vartheta_{\nu})) \\
&= 2\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot \mathcal{D}_{\vartheta_{\mu}} B(\vartheta_{\nu}).
\end{aligned} \quad (506)$$

After some algebraic manipulations the right side of Eq.(506) may be written as

$$\begin{aligned}
\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot \mathcal{D}_{\vartheta_{\mu}} B(\vartheta_{\nu}) &= \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot \vartheta_{\mu} \cdot \partial B(\vartheta_{\nu}) + \vartheta_{\mu} \cdot \partial \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu}) \\
&\quad + \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot (\Omega(\vartheta_{\mu}) \underset{\eta^{-1}}{\times} B(\vartheta_{\nu})) - \vartheta_{\mu} \cdot \partial \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu}) \\
&= \vartheta_{\mu} \cdot \partial(\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu})) - (\vartheta_{\mu} \cdot \partial \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \\
&\quad + \underline{\eta}(\Omega(\vartheta_{\mu}) \underset{\eta}{\times} \underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}))) \cdot B(\vartheta_{\nu}) \\
&= \vartheta_{\mu} \cdot \partial(\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu})) - \underline{\eta}(\vartheta_{\mu} \cdot \partial \eta \underline{\boldsymbol{h}}^{\clubsuit}(\vartheta^{\mu} \wedge \vartheta^{\nu}) \\
&\quad + \Omega(\vartheta_{\mu}) \underset{\eta}{\times} \underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu})) \cdot B(\vartheta_{\nu}),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot \mathcal{D}_{\vartheta_{\mu}} B(\vartheta_{\nu}) &= \vartheta_{\mu} \cdot \partial(\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu})) \\
&\quad - \underline{\eta}\mathcal{D}_{\vartheta_{\mu}} \underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu}).
\end{aligned} \quad (507)$$

Also, the first member on the right side of Eq.(507) may be written as a scalar divergent of a 1-form field, i.e.,

$$\vartheta_{\mu} \cdot \partial(\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu})) = \partial \cdot (\vartheta_{\mu}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu})), \quad (508)$$

where we used the fact that $\partial \cdot (\vartheta_{\mu}\phi) = \vartheta_{\mu} \cdot \partial\phi$, where ϕ is a scalar field.

After putting Eq.(508) in Eq.(507) and using the result obtained in Eq.(506), the Eq.(505) gives finally the variational formula

$$\begin{aligned}
\boldsymbol{\delta}_{\Omega}^{B}(\underline{\boldsymbol{h}}^{\clubsuit}(\partial_a \wedge \partial_b) \cdot \mathcal{R}_2(a \wedge b) \det[\boldsymbol{h}]) &= -2(\underline{\eta}\mathcal{D}_{\vartheta_{\mu}} \underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu}) \\
&\quad - \partial \cdot (\vartheta_{\mu}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^{\mu} \wedge \boldsymbol{\vartheta}^{\nu}) \cdot B(\vartheta_{\nu}))) \det[\boldsymbol{h}].
\end{aligned} \quad (509)$$

Putting Eq.(509) in Eq.(497), the contour problem for the dynamic variable Ω becomes

$$\begin{aligned}&\int_U \underline{\eta}\mathcal{D}_{\vartheta_\mu}\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\vartheta_\nu)\det[\boldsymbol{h}]\;\tau \\ &-\int_U \partial\cdot(\vartheta_\mu \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\vartheta_\nu))\det[\boldsymbol{h}]\;\tau = 0,\end{aligned} \tag{510}$$

for all B, such that $B|_{\partial U} = 0$. Now, the second member of Eq.(510) may be easily transformed in such a way that a scalar divergent of a 1-form field appears. Indeed, if we recall that and $\partial\cdot(a\phi) = a\cdot\partial\phi + (\partial\cdot a)\phi$, where a is a 1-form field and ϕ is a scalar field we have

$$\begin{aligned}\int_U \partial\cdot(\vartheta_\mu \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\vartheta_\nu))\det[\boldsymbol{h}]\;\tau &= \int_U \partial\cdot(\vartheta_\mu \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\vartheta_\nu))\det[\boldsymbol{h}])\;\tau \\ &- \int_U \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\vartheta_\nu)\vartheta_\mu\cdot\partial\det[\boldsymbol{h}]\;\tau.\end{aligned} \tag{511}$$

Putting Eq.(511) in Eq.(510) we get

$$\begin{aligned}&\int_U (\underline{\eta}\mathcal{D}_{\vartheta_\mu}\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\det[\boldsymbol{h}] + \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\vartheta_\mu\cdot\partial\det[\boldsymbol{h}])\cdot B(\vartheta_\nu)\;\tau \\ &\qquad - \int_U \partial\cdot(\vartheta_\mu \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\vartheta_\nu))\det[\boldsymbol{h}])\;\tau = 0,\end{aligned} \tag{512}$$

for all B $B|_{\partial U} = 0$. Utilizing next the Gauss-Stokes theorem with the boundary condition $B|_{\partial U} = 0$, the second term of Eq.(512) may be integrated and gives

$$\begin{aligned}\int_U \partial\cdot(\vartheta_\mu \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\boldsymbol{\vartheta}_\nu))\det[\boldsymbol{h}])\;\tau &= \int_{\partial U} \boldsymbol{\vartheta}^\rho\cdot\vartheta_\mu \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\vartheta_\nu)\det[\boldsymbol{h}]\;\tau_\rho \\ &= \int_{\partial U} \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\cdot B(\vartheta_\nu)\det[\boldsymbol{h}]\;\tau_\mu = 0.\end{aligned} \tag{513}$$

So, putting Eq.(513) in Eq.(512) gives

$$\int_U (\underline{\eta}\mathcal{D}_{\vartheta_\mu}\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\det[\boldsymbol{h}] + \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\vartheta_\mu\cdot\partial\det[\boldsymbol{h}])\cdot B(\vartheta_\nu)\;\tau = 0, \tag{514}$$

for all B such $B|_{\partial U} = 0$, and since B is arbitrary a fundamental lemma of integration theory finally yields

$$\underline{\eta}\mathcal{D}_{\vartheta_\mu}\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\det[\boldsymbol{h}] + \underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu)\vartheta_\mu\cdot\partial\det[\boldsymbol{h}] = 0,$$

i.e.,

$$\mathcal{D}_{\vartheta_\mu}\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu) = -(\vartheta_\mu\cdot\partial\ln|\det[\boldsymbol{h}]|)\underline{\eta}\underline{\boldsymbol{h}}^{*}(\boldsymbol{\vartheta}^\mu \wedge \boldsymbol{\vartheta}^\nu), \tag{515}$$

or yet,

$$(\vartheta_\mu\cdot\partial\ln|\det[\boldsymbol{h}]|)\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)+\vartheta_\mu\cdot\partial\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)+\Omega(\vartheta_\mu)\underset{\eta}{\times}\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)=0 \tag{516}$$

This is the field equation for the $\varkappa\eta$-gauge rotation field Ω. It resumes a set of four coupled multiform equations, differential in the canonical 1-forms of $\boldsymbol{h}^{\clubsuit}$ (i.e., $\boldsymbol{h}^{\clubsuit}(\boldsymbol{\vartheta}^0),\ldots,\boldsymbol{h}^{\clubsuit}(\boldsymbol{\vartheta}^3)$) and algebraic in the canonical biforms of Ω (i.e., $\Omega(\vartheta_0),\ldots,\Omega(\vartheta_3)$).

We end this Appendix, writing again the Euler-Lagrange equations for the fields $\boldsymbol{h}$ and Ω, in terms of the canonical 1-form and biform fields.

$$\boldsymbol{h}^{\clubsuit}(\boldsymbol{\vartheta}^\mu)\lrcorner(\vartheta_\mu\cdot\partial\Omega(\vartheta_\nu)-\vartheta_\nu\cdot\partial\Omega(\vartheta_\mu)+\Omega(\vartheta_\mu)\underset{\eta}{\times}\Omega(\vartheta_\nu))=0,$$

$$(\vartheta_\mu\cdot\partial\ln|\det[\boldsymbol{h}]|)\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)+\vartheta_\mu\cdot\partial\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)+\Omega(\vartheta_\mu)\underset{\eta}{\times}\underline{\eta}\underline{\boldsymbol{h}}^{\clubsuit}(\boldsymbol{\vartheta}^\mu\wedge\boldsymbol{\vartheta}^\nu)=0.$$

D Proof of Eq.(352)

Proposition[82] The Einstein-Hilbert Lagrangian density L_{eh} can be written as

$$\mathcal{L}_{eh}=-d\left(\mathfrak{g}^\alpha\wedge\underset{g}{\star}d\mathfrak{g}_\alpha\right)+\mathcal{L}_g, \tag{517}$$

where

$$\mathcal{L}_g=-\frac{1}{2}d\mathfrak{g}^\alpha\wedge\underset{g}{\star}d\mathfrak{g}_\alpha+\frac{1}{2}\underset{g}{\delta}\mathfrak{g}^\alpha\wedge\underset{g}{\star}\underset{g}{\delta}\mathfrak{g}_\alpha+\frac{1}{4}d\mathfrak{g}^\alpha\wedge\mathfrak{g}_\alpha\wedge\underset{g}{\star}(d\mathfrak{g}^\alpha\wedge\mathfrak{g}_\alpha) \tag{518}$$

is the first order Lagrangian density (first introduced by Einstein).

Proof We will establish that $\mathcal{L}_g$ may be written as

$$\mathcal{L}_g=-\frac{1}{2}\tau_g\mathfrak{g}^\gamma\underset{g^{-1}}{\lrcorner}\mathfrak{g}^\beta\underset{g^{-1}}{\lrcorner}\left(\omega_{\alpha\beta}\wedge\omega_\gamma^\alpha\right), \tag{519}$$

which is nothing more than the *intrinsic* form of the Einstein first order Lagrangian in the gauge defined by the basis $\{\mathfrak{g}^\alpha\}$. To see this first recall Cartan's structure equations for the Lorentzian structure $(M\simeq\mathbb{R}^4,\boldsymbol{g},D)$, i.e.,

$$d\mathfrak{g}^\alpha=-\omega_\beta^\alpha\wedge\mathfrak{g}^\beta, \tag{520a}$$

$$\mathcal{R}_\beta^\alpha=d\omega_\beta^\alpha+\omega_\gamma^\alpha\wedge\omega_\beta^\gamma, \tag{520b}$$

where $\omega_{\mathbf{b}}^{\mathbf{a}}$ are the so called connection 1-form fields, which on the basis $\{\mathfrak{g}^{\mathbf{a}}\}$ is written as

$$\omega_\beta^\alpha=L_{\beta\gamma}^\alpha\mathfrak{g}^\gamma. \tag{521}$$

Now, using one of the identities in Eq.(93) we easily get

$$\mathfrak{g}^\gamma\underset{g^{-1}}{\lrcorner}\mathfrak{g}^\beta\underset{g^{-1}}{\lrcorner}\left(\omega_{\alpha\beta}\wedge\omega_\gamma^\alpha\right)=\eta^{\beta\kappa}\left(L_{\kappa\gamma}^\delta L_{\delta\beta}^\gamma-L_{\delta\gamma}^\delta L_{\kappa\beta}^\gamma\right). \tag{522}$$

Next we recall that Eq.(520a) can easily be solved for $\omega_{\mathbf{b}}^{\mathbf{a}}$ in terms of the $d\mathfrak{g}^{\mathbf{a}}$'s and the $\mathfrak{g}^{\mathbf{b}}$'s. We get

$$\omega^{\gamma\delta} = \frac{1}{2}\left[\mathfrak{g}^{\delta} \underset{g^{-1}}{\lrcorner} d\mathfrak{g}^{\gamma} - \mathfrak{g}^{\gamma} \underset{g^{-1}}{\lrcorner} d\mathfrak{g}^{\delta} + \mathfrak{g}^{\gamma} \underset{g^{-1}}{\lrcorner} \left(\mathfrak{g}^{\delta} \underset{g^{-1}}{\lrcorner} d\mathfrak{g}_{\alpha}\right) \mathfrak{g}^{\alpha}\right] \tag{523}$$

Proof. We are now prepared to prove that $\mathcal{L}_{eh}$ can be written as in Eq.(517). We start using Cartan's second structure equation (Eq.(520b)) to write Eq.(350) as:

$$\begin{aligned}
\mathcal{L}_{eh} &= \frac{1}{2}d\omega_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta}) + \frac{1}{2}\omega_{\alpha\gamma} \wedge \omega_{\beta}^{\gamma} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta}) \\
&= \frac{1}{2}d[\omega_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta})] + \frac{1}{2}\omega_{\alpha\beta} \wedge \underset{g}{\star}d(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta}) + \frac{1}{2}\omega_{\alpha\gamma} \wedge \omega_{\beta}^{\gamma} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta}) \\
&= \frac{1}{2}d[\omega_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta})] - \frac{1}{2}\omega_{\alpha\beta} \wedge \omega_{\gamma}^{\alpha} \wedge \underset{g}{\star}(\mathfrak{g}^{\gamma} \wedge \mathfrak{g}^{\beta}).
\end{aligned} \tag{524}$$

Next using again some of the identities in Eq.(111) we get (recall that $\omega^{\gamma\delta} = -\omega^{\delta\gamma}$)

$$\begin{aligned}
(\mathfrak{g}^{\gamma} \wedge \mathfrak{g}^{\delta}) \wedge \underset{g}{\star}\omega_{\gamma\delta} &= -\underset{g}{\star}[\omega^{\gamma\delta} \underset{g}{\lrcorner}(\mathfrak{g}_{\gamma} \wedge \mathfrak{g}_{\delta})] \\
&= \underset{g}{\star}[(\omega^{\gamma\delta} \underset{g}{\cdot} \mathfrak{g}_{\delta})\theta_{\gamma} - (\omega^{\gamma\delta} \underset{g}{\cdot} \mathfrak{g}_{\gamma})\mathfrak{g}_{\delta}] = 2\underset{g}{\star}[(\omega^{\gamma\delta} \underset{g}{\cdot} \mathfrak{g}_{\delta})\mathfrak{g}_{\gamma}],
\end{aligned} \tag{525}$$

and from Cartan's first structure equation (Eq.(520a)) we have

$$\begin{aligned}
\mathfrak{g}^{\alpha} \wedge \underset{g}{\star}d\mathfrak{g}_{\alpha} &= \mathfrak{g}^{\alpha} \wedge \underset{g}{\star}(\omega_{\beta\alpha} \wedge \mathfrak{g}^{\beta}) = -\underset{g}{\star}[\mathfrak{g}_{\alpha} \underset{g}{\lrcorner}(\omega^{\beta\alpha} \wedge \mathfrak{g}_{\beta})] \\
&= -\underset{g}{\star}[\mathfrak{g}_{\alpha} \underset{g}{\cdot} \omega^{\beta\alpha})\mathfrak{g}_{\beta}] = -\underset{g}{\star}[(\mathfrak{g}^{\alpha} \underset{g}{\cdot} \omega_{\beta\alpha})\mathfrak{g}^{\beta}],
\end{aligned} \tag{526}$$

from where it follows that

$$\frac{1}{2}d[(\mathfrak{g}^{\gamma} \wedge \mathfrak{g}^{\delta}) \wedge \underset{g}{\star}\omega_{\gamma\delta}] = -d(\mathfrak{g}^{\alpha} \wedge \underset{g}{\star}d\mathfrak{g}_{\alpha}). \tag{527}$$

On the other hand the second term in the last line of Eq.(524) can be written as

$$\begin{aligned}
&\frac{1}{2}\omega_{\alpha\beta} \wedge \omega_{\gamma}^{\alpha} \wedge \underset{g}{\star}(\mathfrak{g}^{\gamma} \wedge \mathfrak{g}^{\beta}) \\
&= -\frac{1}{2}\underset{g}{\star}[(\mathfrak{g}^{\beta} \underset{g}{\cdot} \omega_{\alpha\beta})(\mathfrak{g}^{\gamma} \underset{g}{\cdot} \omega_{\gamma}^{\alpha}) - (\mathfrak{g}^{\beta} \underset{g}{\cdot} \omega_{\gamma}^{\alpha})(\mathfrak{g}^{\gamma} \underset{g}{\cdot} \omega_{\alpha\beta})].
\end{aligned}$$

Now,

$$\begin{aligned}
&(\mathfrak{g}^{\beta} \underset{g}{\cdot} \omega_{\alpha\beta})(\mathfrak{g}^{\gamma} \underset{g}{\cdot} \omega_{\gamma}^{\alpha}) \\
&= \omega_{\alpha\beta} \underset{g}{\cdot} [(\mathfrak{g}^{\gamma} \underset{g}{\cdot} \omega_{\gamma}^{\alpha})\mathfrak{g}^{\beta}] \\
&= \omega_{\alpha\beta} \underset{g}{\cdot} [\mathfrak{g}^{\gamma} \underset{g}{\lrcorner}(\omega_{\gamma}^{\alpha} \wedge \mathfrak{g}^{\beta}) + \omega^{\alpha\beta}] \\
&= (\omega_{\alpha\beta} \wedge \mathfrak{g}^{\gamma}) \underset{g}{\lrcorner}(\omega_{\gamma}^{\alpha} \wedge \mathfrak{g}^{\beta}) + \omega_{\gamma\delta} \underset{g}{\cdot} \omega^{\gamma\delta}]
\end{aligned}$$

and taking into account that $d\mathfrak{g}^\alpha = -\omega^\alpha_\beta \wedge \mathfrak{g}^\beta$, $\underset{g}{\star}^{-1} d \underset{g}{\star} \mathfrak{g}^\alpha = -\omega^\alpha_\beta \underset{g}{\cdot} \mathfrak{g}^\beta$ and that $\underset{g}{\delta}\mathfrak{g}_\alpha = -\underset{g}{\star}^{-1} d \underset{g}{\star} \mathfrak{g}_\alpha$ we have

$$\frac{1}{2}\omega_{\alpha\beta} \wedge \omega^\alpha_\gamma \wedge \underset{g}{\star}(\mathfrak{g}^\gamma \wedge \mathfrak{g}^\beta) = \frac{1}{2}[-d\mathfrak{g}^\alpha \wedge \underset{g}{\star} d\mathfrak{g}_\alpha - \underset{g}{\delta}\mathfrak{g}^\alpha \wedge \underset{g}{\star}\underset{g}{\delta}\mathfrak{g}_\alpha + \omega_{\gamma\delta} \wedge \underset{g}{\star}\omega^{\gamma\delta}] \tag{528}$$

Next, using Eq.(523) the last term in the last equation can after some algebra be written as

$$\frac{1}{2}\omega_{\gamma\delta} \wedge \underset{g}{\star}\omega^{\gamma\delta} = d\mathfrak{g}^\alpha \wedge \underset{g}{\star} d\mathfrak{g}_\alpha - \frac{1}{4}\left(d\mathfrak{g}^\alpha \wedge \mathfrak{g}_\alpha\right) \wedge \underset{\mathfrak{g}}{\star}\left(d\mathfrak{g}^\alpha \wedge \mathfrak{g}_\alpha\right)$$

and we finally get

$$\mathcal{L}_g = -\frac{1}{2} d\mathfrak{g}^\alpha \wedge \underset{g}{\star} d\mathfrak{g}_\alpha + \frac{1}{2}\underset{g}{\delta}\mathfrak{g}^\alpha \wedge \underset{g}{\star}\underset{g}{\delta}\mathfrak{g}_\alpha + \frac{1}{4}\left(d\mathfrak{g}^\alpha \wedge \mathfrak{g}_\alpha\right) \wedge \underset{g}{\star}\left(d\mathfrak{g}^\beta \wedge \mathfrak{g}_\beta\right), \tag{529}$$

and the proposition is proved. ■

E Derivation of the Field Equations from the Einstein-Hilbert Lagrangian Density

Let $X \in \sec \bigwedge^p T^*U$. *A multiform functional* F of X (not depending *explicitly* on $x \in \bigwedge^1 U$) is a mapping

$$F : \sec \bigwedge^p T^*U \to \sec \bigwedge^r T^*U$$

As, in Section 3, when no confusion arises we use a sloppy notation and denote the image $F(X) \in \sec \bigwedge^r T^*U$ simply by F. Eventually we also denote a functional F by $F(X)$. Which object we are talking about is always obvious from the context of the equations where they appear.

Let $w := \boldsymbol{\delta} X \in \sec \bigwedge^p T^*U$. As in Exercise 3.6. we write the *variation* of F in the direction of $\boldsymbol{\delta} X$ as the functional $\boldsymbol{\delta} F : \bigwedge^p U \to \bigwedge^r U$ given by

$$\boldsymbol{\delta} F = \lim_{\lambda \to 0} \frac{F(X + \lambda \boldsymbol{\delta} X) - F(X)}{\lambda}. \tag{530}$$

Recall from Remark 3.1 that the *algebraic derivative* of F relative to X, $\frac{\partial F}{\partial X}$ is such that:

$$\boldsymbol{\delta} F = \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X}. \tag{531}$$

Moreover, given the functionals $F : \bigwedge^p U \to \bigwedge^p U \to$ and $G : \bigwedge^p U \to \bigwedge^s U$ the variation $\boldsymbol{\delta}$ satisfies

$$\boldsymbol{\delta}(F \wedge G) = \boldsymbol{\delta} F \wedge G + F \wedge \boldsymbol{\delta} G, \tag{532}$$

and the algebraic derivative (as it is trivial to verify) satisfies

$$\frac{\partial}{\partial X}(F \wedge G) = \frac{\partial F}{\partial X} \wedge G + (-1)^{rp} F \wedge \frac{\partial G}{\partial X}. \tag{533}$$

An important property of $\boldsymbol{\delta}$ is that it commutes with the exterior derivative operator d, i.e., for any given functional F

$$d\boldsymbol{\delta} F = \boldsymbol{\delta} dF. \tag{534}$$

In general we may have functionals depending on several different multiform forms fields, say, $F : \sec(\bigwedge^p T^*U \times \bigwedge^q T^*U) \rightarrow \sec \bigwedge^r T^*U$, with $(X, Y) \mapsto F(X, Y) \in \sec \bigwedge^p T^*U$. In this case we have:

$$\boldsymbol{\delta} F = \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} + \boldsymbol{\delta} Y \wedge \frac{\partial F}{\partial Y}. \tag{535}$$

An important case is the one where the functional F is such that $F(X, dX) : \sec(\bigwedge^p T^*U \times \bigwedge^{p+1} T^*U) \rightarrow \sec \bigwedge^4 T^*U$. We then can write, supposing that the variation $\boldsymbol{\delta} X$ is chosen to be null in the boundary $\partial U^{'}$, $U' \subset U$ (or that $\frac{\partial F}{\partial dX}\big|_{\partial U'} = 0$) and taking into account Stokes theorem,

$$\begin{aligned}
\boldsymbol{\delta} \int_{U'} F &= \int_{U'} \boldsymbol{\delta} F = \int_{U'} \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial X} + \boldsymbol{\delta} dX \wedge \frac{\partial F}{\partial dX} \\
&= \int_{U'} \boldsymbol{\delta} X \wedge \left[\frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right)\right] + d\left(\boldsymbol{\delta} X \wedge \frac{\partial F}{\partial dX}\right) \\
&= \int_{U'} \boldsymbol{\delta} X \wedge \left[\frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right)\right] + \int_{\partial U'} \boldsymbol{\delta} X \wedge \frac{\partial F}{\partial dX} \\
&= \int_{U'} \boldsymbol{\delta} X \wedge \frac{\boldsymbol{\delta} F}{\boldsymbol{\delta} X},
\end{aligned} \tag{536}$$

where $\frac{\boldsymbol{\delta}}{\boldsymbol{\delta} X} F(X, dX) : \sec(\bigwedge^p T^*U \times \bigwedge^{p+1} T^*U) \rightarrow \bigwedge^{4-p} T^*U$ is called the *functional derivative* of F and we have:

$$\frac{\boldsymbol{\delta} F}{\boldsymbol{\delta} X} = \frac{\partial F}{\partial X} - (-1)^p d\left(\frac{\partial F}{\partial dX}\right). \tag{537}$$

When $F = \mathcal{L}$ is a Lagrangian density in field theory $\frac{\boldsymbol{\delta}\mathcal{L}}{\boldsymbol{\delta} X}$ is the *Euler-Lagrange functional*.

We now obtain the variation of the Einstein-Hilbert Lagrangian density $\mathcal{L}_{eh}$ given by Eq.(350), i.e.,

$$\mathcal{L}_{eh} = \frac{1}{2}(\mathfrak{g}^\kappa \wedge \mathfrak{g}^\iota) \wedge \underset{g}{\star} \mathcal{R}_{\kappa\iota} = \frac{1}{2}\mathcal{R}_{\kappa\iota} \wedge \underset{g}{\star}(\mathfrak{g}^\kappa \wedge \mathfrak{g}^\iota),$$

by varying the $\omega_{\gamma\kappa}$ and the $\mathfrak{g}^\kappa$ independently.

We have[69]

$$\begin{aligned}\boldsymbol{\delta}\mathcal{L}_{eh} &= \frac{1}{2}\boldsymbol{\delta}[\mathcal{R}_{\gamma\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})] \\ &= \frac{1}{2}\boldsymbol{\delta}\mathcal{R}_{\gamma\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) + \frac{1}{2}\mathcal{R}_{\gamma\delta}\wedge\boldsymbol{\delta}\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}). \end{aligned} \tag{538}$$

From Cartan's second structure equation we can write

$$\begin{aligned}&\boldsymbol{\delta}\mathcal{R}_{\gamma\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) \\ &= \boldsymbol{\delta}d\omega_{\gamma\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) + \boldsymbol{\delta}\omega_{\gamma\kappa}\wedge\omega_{\delta}^{\kappa}\wedge\star(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) + \omega_{\gamma\kappa}\wedge\boldsymbol{\delta}\omega_{\delta}^{\kappa}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) \\ &= \boldsymbol{\delta}d\omega_{\gamma\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) \qquad (539) \\ &= d[\boldsymbol{\delta}\omega_{\gamma\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})] - \boldsymbol{\delta}\omega_{\gamma\delta}\wedge d[\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})]. \\ &= d[\boldsymbol{\delta}\omega_{\gamma\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})] - \boldsymbol{\delta}\omega_{\gamma\delta}\wedge[-\omega_{\kappa}^{\alpha}\wedge \underset{g}{\star}(\mathfrak{g}^{\kappa}\wedge\mathfrak{g}^{\kappa}) - \omega_{\kappa}^{\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\kappa})] \\ &= d[\boldsymbol{\delta}\omega_{\gamma\delta}\wedge \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})].\end{aligned}$$

Moreover, using the definition of algebraic derivative (Eq.(530)) we have immediately

$$\boldsymbol{\delta}\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) = \boldsymbol{\delta}\mathfrak{g}^{\kappa}\wedge\frac{\partial[\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})]}{\partial\mathfrak{g}^{\kappa}} \tag{540}$$

Now recalling Eq.(112) of Section 4 we can write

$$\begin{aligned}\boldsymbol{\delta}\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) &= \boldsymbol{\delta}(\frac{1}{2}\eta^{\gamma\kappa}\eta^{\delta\iota}\epsilon_{\kappa\iota\mu\nu}\mathfrak{g}^{\mu}\wedge\mathfrak{g}^{\nu}) \\ &= \boldsymbol{\delta}\mathfrak{g}^{\mu}\wedge(\eta^{\gamma\kappa}\eta^{\delta\iota}\epsilon_{\kappa\iota\mu\nu}\mathfrak{g}^{\nu}),\end{aligned} \tag{541}$$

from where we get

$$\frac{\partial\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})}{\partial\mathfrak{g}^{\mathbf{m}}} = \eta^{\gamma\kappa}\eta^{\delta\iota}\epsilon_{\kappa\iota\mu\nu}\mathfrak{g}^{\nu}. \tag{542}$$

On the other hand we have

$$\begin{aligned}&\mathfrak{g}_{\mu}\lrcorner\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) = \mathfrak{g}_{\mu}\underset{g}{\lrcorner}(\frac{1}{2}\eta^{\gamma\kappa}\eta^{\delta\iota}\epsilon_{\kappa\iota\rho\sigma}\mathfrak{g}^{\rho}\wedge\mathfrak{g}^{\sigma}) \\ &= \eta^{\gamma\kappa}\eta^{\delta\iota}\epsilon_{\kappa\iota\mu\nu}\mathfrak{g}^{\nu}.\end{aligned} \tag{543}$$

Moreover, using the fourth formula in Eq.(111), we can write

$$\begin{aligned}\frac{\partial[\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})]}{\partial\mathfrak{g}^{\kappa}} &= \mathfrak{g}_{\kappa}\underset{g}{\lrcorner}\underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}) \\ &= \underset{g}{\star}[\mathfrak{g}_{\kappa}\wedge(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta})] = \underset{g}{\star}(\mathfrak{g}^{\gamma}\wedge\mathfrak{g}^{\delta}\wedge\mathfrak{g}_{\kappa}).\end{aligned} \tag{544}$$

[69]We use only constrained variations of the $\mathfrak{g}^{\alpha}$ that do no change the metric field $\boldsymbol{g}$.

Finally,

$$\boldsymbol{\delta} \underset{g}{\star} (\mathfrak{g}^\gamma \wedge \mathfrak{g}^\delta) = \boldsymbol{\delta}\mathfrak{g}^\kappa \wedge \underset{g}{\star}(\mathfrak{g}^\gamma \wedge \mathfrak{g}^\delta \wedge \mathfrak{g}_\kappa). \tag{545}$$

Then using Eq.(539) and Eq.(545) in Eq.(538) we get

$$\boldsymbol{\delta}\mathcal{L}_{eh} = \frac{1}{2} d[\boldsymbol{\delta}\omega_{\gamma\delta} \wedge \underset{g}{\star}(\mathfrak{g}^\gamma \wedge \mathfrak{g}^\delta)] + \boldsymbol{\delta}\mathfrak{g}^\kappa \wedge [\frac{1}{2}\mathcal{R}_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^\alpha \wedge \mathfrak{g}^\beta \wedge \mathfrak{g}_\kappa)]. \tag{546}$$

Now, the curvature 2-form fields are given by $\mathcal{R}_{\alpha\beta} = \frac{1}{2} R_{\alpha\beta\gamma\kappa}\mathfrak{g}^\gamma \wedge \mathfrak{g}^\kappa$ where $R_{\alpha\beta\gamma\kappa}$ are the components of the Riemann tensor and then

$$\begin{aligned}
\frac{1}{2}\mathcal{R}_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^\alpha \wedge \mathfrak{g}^\beta \wedge \mathfrak{g}_\mu) &= -\frac{1}{2} \underset{g}{\star} [\mathcal{R}_{\alpha\beta} \underset{g}{\lrcorner} (\mathfrak{g}^\alpha \wedge \mathfrak{g}^\beta \wedge \mathfrak{g}_\mu)] \\
&= -\frac{1}{4} R_{\alpha\beta\gamma\kappa} \underset{g}{\star} [(\mathfrak{g}^\gamma \wedge \mathfrak{g}^\kappa) \underset{g}{\lrcorner} (\mathfrak{g}^\alpha \wedge \mathfrak{g}^\beta \wedge \mathfrak{g}_\mu)] \\
&= - \star (\mathcal{R}_\mu - \frac{1}{2} R \mathfrak{g}_\mu) = - \star \mathcal{G}_\mu,
\end{aligned} \tag{547}$$

where the $\mathcal{R}_\mu$ are the Ricci 1-form fields and $\mathcal{G}_\mu$ are the Einstein 1-form fields. So, finally, we have and so we can write

$$\int \boldsymbol{\delta}(\mathcal{L}_{eh} + \mathcal{L}_m) = \int \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge (- \star \mathcal{G}_\alpha + \frac{\partial \mathfrak{L}_m}{\partial \mathfrak{g}^\alpha}) = 0. \tag{548}$$

To have the equations of motion in the form of Eq.(363) (here with $m = 0$), i.e.,

$$d \underset{g}{\star} \mathcal{S}^\gamma + \underset{g}{\star} t^\gamma = - \underset{g}{\star} \mathcal{T}^\gamma \tag{549}$$

it remains to prove that

$$d \underset{g}{\star} \mathcal{S}^\gamma + \underset{g}{\star} t^\gamma = - \star \mathcal{G}^\gamma. \tag{550}$$

In order to do that we recall that $\mathcal{L}_{eh}$ may be written (recall Eq.(352) and Eq.(356)) as

$$\begin{aligned}
\mathcal{L}_{eh} &= \mathcal{L}_g - d(\mathfrak{g}^\alpha \wedge \underset{g}{\star} d\mathfrak{g}_\beta) \\
&= -\frac{1}{2} d\mathfrak{g}^\alpha \wedge \underset{g}{\star} d\mathfrak{g}_\alpha + \frac{1}{2} \underset{g}{\delta}\mathfrak{g}^\alpha \wedge \underset{g}{\star}\underset{g}{\delta}\mathfrak{g}_\alpha + \frac{1}{4} d\mathfrak{g}^\alpha \wedge \mathfrak{g}_\alpha \wedge \underset{g}{\star}(d\mathfrak{g}^\beta \wedge \mathfrak{g}_\beta) - d(\mathfrak{g}^\alpha \wedge \underset{g}{\star} d\mathfrak{g}_\beta) \\
&= -\frac{1}{2} d\mathfrak{g}^\alpha \wedge \mathfrak{g}_\beta \wedge \underset{g}{\star}(d\mathfrak{g}^\beta \wedge \mathfrak{g}_\alpha) + \frac{1}{4} d\mathfrak{g}^\alpha \wedge \mathfrak{g}_\alpha \wedge \underset{g}{\star}(d\mathfrak{g}^\beta \wedge \mathfrak{g}_\beta) - d(\mathfrak{g}^\alpha \wedge \underset{g}{\star} d\mathfrak{g}_\beta) \\
&= \frac{1}{2} d\mathfrak{g}_\alpha \wedge \underset{g}{\star}\mathcal{S}^\alpha - d(\mathfrak{g}^\alpha \wedge \underset{g}{\star} d\mathfrak{g}_\beta)
\end{aligned} \tag{551}$$

Then

$$\begin{aligned}
\boldsymbol{\delta}\mathcal{L}_g &= \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\partial \mathcal{L}_g}{\partial \mathfrak{g}^\alpha} + \boldsymbol{\delta} d\mathfrak{g}^\alpha \wedge \frac{\partial \mathcal{L}_g}{\partial d\mathfrak{g}^\alpha} \\
& \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \left[\frac{\partial \mathcal{L}_g}{\partial \mathfrak{g}^\alpha} + d\left(\frac{\partial \mathcal{L}_g}{\partial d\mathfrak{g}^\alpha}\right)\right] + d\left(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \frac{\partial \mathcal{L}_g}{\partial d\mathfrak{g}^\alpha}\right) \\
&= \boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \left(\underset{g}{\star} t_\alpha + d \underset{g}{\star} \mathcal{S}_\alpha\right) + d(\boldsymbol{\delta}\mathfrak{g}^\alpha \wedge \underset{g}{\star}\mathcal{S}_\alpha),
\end{aligned} \tag{552}$$

with

$$\underset{g}{\star}\, t_\alpha := \frac{\partial \mathcal{L}_g}{\partial \mathfrak{g}^\alpha}, \quad \underset{g}{\star}\, \mathcal{S}_\alpha := \frac{\partial \mathcal{L}_g}{\partial d\mathfrak{g}^\alpha} \tag{553}$$

To calculate $\frac{\partial \mathfrak{L}_g}{\partial \mathfrak{g}^\alpha}$ we first recall from Eq.(32) that we can write

$$\mathfrak{g}_\kappa \underset{g}{\lrcorner} (\frac{1}{2} d\mathfrak{g}_\alpha \wedge \underset{g}{\star} \mathcal{S}^\alpha) = \frac{1}{2}(\mathfrak{g}_\kappa \underset{g}{\lrcorner} d\mathfrak{g}_\alpha) \wedge \underset{g}{\star}\mathcal{S}^\alpha + \frac{1}{2} d\mathfrak{g}_\alpha \wedge (\mathfrak{g}_\kappa \underset{g}{\lrcorner} \underset{g}{\star}\, \mathcal{S}^\alpha) \tag{554}$$

and thus since from Eq.(551) it is $\mathcal{L}_g = \frac{1}{2} d\mathfrak{g}_\alpha \wedge \underset{g}{\star}\mathcal{S}^\alpha$, we have

$$\mathfrak{g}_\kappa \underset{g}{\lrcorner} \mathcal{L}_g - (\mathfrak{g}_\kappa \underset{g}{\lrcorner} d\mathfrak{g}_\alpha) \wedge \underset{g}{\star}\mathcal{S}^\alpha = \frac{1}{2}\left[d\mathfrak{g}_\alpha \wedge (\mathfrak{g}_\kappa \underset{g}{\lrcorner} \underset{g}{\star}\, \mathcal{S}^\alpha) - (\mathfrak{g}_\kappa \underset{g}{\lrcorner} d\mathfrak{g}_\alpha) \wedge \underset{g}{\star}\mathcal{S}^\alpha \right] \tag{555}$$

Next we recall that

$$\begin{aligned} \boldsymbol{\delta}\mathcal{S}^\alpha &= \frac{1}{2}\mathcal{S}^\alpha_{\kappa\iota}\boldsymbol{\delta}(\mathfrak{g}^\kappa \wedge \mathfrak{g}^\iota) = \boldsymbol{\delta}\mathfrak{g}^\kappa \wedge (\mathfrak{g}_\kappa \underset{g}{\lrcorner} \mathcal{S}^\alpha), \\ \boldsymbol{\delta} \underset{g}{\star}\, \mathcal{S}^\alpha &= \boldsymbol{\delta}\mathfrak{g}^\kappa \wedge (\mathfrak{g}_\kappa \underset{g}{\lrcorner} \underset{g}{\star}\, \mathcal{S}^\alpha) \end{aligned} \tag{556}$$

and so we have

$$\begin{aligned} \left[\boldsymbol{\delta}, \underset{g}{\star}\right] \mathcal{S}^\alpha &= \boldsymbol{\delta} \underset{g}{\star}\, \mathcal{S}^\alpha - \underset{g}{\star}\boldsymbol{\delta}\mathcal{S}^\alpha \\ &= \boldsymbol{\delta}\mathfrak{g}^\kappa \wedge (\mathfrak{g}_\kappa \underset{g}{\lrcorner} \mathcal{S}^\alpha) - \boldsymbol{\delta}\mathfrak{g}^\kappa \wedge (\mathfrak{g}_\kappa \underset{g}{\lrcorner} \underset{g}{\star}\, \mathcal{S}^\alpha). \end{aligned} \tag{557}$$

Multplying Eq.(557) on the left by $d\mathfrak{g}_\alpha \wedge$ and moreover adding $\boldsymbol{\delta} d\mathfrak{g}^\kappa \wedge \underset{g}{\star}\mathcal{S}^\alpha$ to both sides we get

$$\begin{aligned} \boldsymbol{\delta}(\frac{1}{2}\mathfrak{g}^\kappa \wedge \underset{g}{\star}\mathcal{S}^\alpha) &= \boldsymbol{\delta} d\mathfrak{g}^\kappa \wedge \frac{1}{2} \underset{g}{\star}\, \mathcal{S}^\alpha + \frac{1}{2} d\mathfrak{g}^\kappa \wedge \underset{g}{\star}\boldsymbol{\delta}\mathcal{S}^\alpha \\ &\quad + \boldsymbol{\delta}\mathfrak{g}^\kappa \wedge \frac{1}{2}\left(d\mathfrak{g}_\alpha \wedge (\mathfrak{g}_\kappa \underset{g}{\lrcorner} \underset{g}{\star}\, \mathcal{S}^\alpha) - (\mathfrak{g}_\kappa \underset{g}{\lrcorner} d\mathfrak{g}_\alpha) \wedge \underset{g}{\star}\mathcal{S}^\alpha \right) \end{aligned}$$

or

$$\begin{aligned} \boldsymbol{\delta}\mathcal{L}_g &= \boldsymbol{\delta} d\mathfrak{g}^\kappa \wedge \frac{1}{2} \underset{g}{\star}\, \mathcal{S}^\alpha + \frac{1}{2} d\mathfrak{g}^\kappa \wedge \underset{g}{\star}\boldsymbol{\delta}\mathcal{S}^\alpha \\ &\quad + \boldsymbol{\delta}\mathfrak{g}^\kappa \wedge \frac{1}{2}\left(d\mathfrak{g}_\alpha \wedge (\mathfrak{g}_\kappa \underset{g}{\lrcorner} \underset{g}{\star}\, \mathcal{S}^\alpha) - (\mathfrak{g}_\kappa \underset{g}{\lrcorner} d\mathfrak{g}_\alpha) \wedge \underset{g}{\star}\mathcal{S}^\alpha \right). \end{aligned} \tag{558}$$

Thus comparing the coefficient of $\boldsymbol{\delta}\mathfrak{g}^\kappa$ in Eq.(558) with the one appearing at the first line in Eq.(552) and taking into account the identity given by Eq.(555) we get

$$\underset{g}{\star} t_\alpha := \tfrac{\partial \mathcal{L}_g}{\partial \mathfrak{g}^\alpha} = \mathfrak{g}_\alpha \underset{g}{\lrcorner} \mathfrak{L}_g - (\mathfrak{g}_\alpha \underset{g}{\lrcorner} d\mathfrak{g}^\kappa) \wedge \underset{g}{\star}\mathcal{S}_\kappa. \tag{559}$$

Also take into account that

$$\underset{g}{\star}\mathcal{S}_\alpha := \tfrac{\partial \mathcal{L}_g}{\partial d\mathfrak{g}^\alpha} = -\mathfrak{g}_\kappa \wedge \underset{g}{\star}(d\mathfrak{g}^\kappa \wedge \mathfrak{g}_\alpha) + \tfrac{1}{2}\mathfrak{g}_\alpha \wedge \underset{g}{\star}(d\mathfrak{g}^\kappa \wedge \mathfrak{g}_\kappa), \tag{560}$$

or

$$\underset{g}{\star}\mathcal{S}^{\kappa} = -\underset{g}{\star}d\mathfrak{g}^{\kappa} - (\mathfrak{g}^{\kappa}\underset{g}{\lrcorner}\underset{g}{\star}\mathfrak{g}^{\alpha}) \wedge \underset{g}{\star}d\underset{g}{\star}\mathfrak{g}_{\alpha} + \frac{1}{2}\mathfrak{g}^{\kappa} \wedge \underset{g}{\star}(d\mathfrak{g}^{\alpha} \wedge \mathfrak{g}_{\alpha}). \tag{561}$$

Finally comparing the coefficient of $\boldsymbol{\delta}\mathfrak{g}^{\kappa}$in Eq.(558) with the one appearing on Eq.(546) we get the proof of Eq.(550).

Remark F1 Before proceeding recall that in our theory the $\star t^{\mathbf{c}}$ are 3-form fields defined directly as functions of the gravitational potentials $\mathfrak{g}^{\alpha}$ and the fields $d\mathfrak{g}^{\alpha}$ (Eqs.(364) and (365)), where the $\{\mathfrak{g}^{\alpha}\}$ define by chance a basis for the $\boldsymbol{g}$-orthonormal bundle. However, from Remark 6.4 we know that the cotetrad fields $\{\mathfrak{g}^{\alpha}\}$ defining the gravitational potentials are associated to the extensor field $\boldsymbol{h}$ modulus an arbitrary local Lorentz rotation Λ which is hidden in the definition of $\boldsymbol{g} = \boldsymbol{h}^{\dagger}\eta\boldsymbol{h} = \boldsymbol{h}^{\dagger}\Lambda^{\dagger}\eta\Lambda\boldsymbol{h}$. Thus the energy-momentum 3-forms $\underset{g}{\star}t^{\gamma}$ are gauge dependent. However, once a gauge is chosen the $\star t^{\gamma}$ are, of course, independent on the basis (coordinate or otherwise) where they are expressed, i.e., they are legitimate tensor fields. However, in *GRT* (where the $\underset{g}{\star}\mathcal{S}^{\gamma} : U \to \bigwedge^2 U$ are called *superpotentials*[70]) and the $\star t^{\gamma}$ (called the *gravitational energy-momentum pseudo* 3-*forms*) this is *not* the case The reason is implicit in the name *pseudo* 3- *forms*. for $\star t^{\gamma}$. Indeed, in *GRT* those objects are directly associated to the connection 1-forms $\omega_{\alpha\beta}$ associated to a particular section of the $\mathbf{g}$-orthonormal bundle and thus besides being gauge dependent, their components in any basis do *not* define a true tensor field.

Taking into account Remark F1 we provide yet alternative expressions for $\underset{g}{\star}t^{\gamma}$ and $\underset{g}{\star}\mathcal{S}^{\gamma}$ given by Eqs.(559) and (560) as

$$\begin{aligned}
\underset{g}{\star}t^{\gamma} &= -\frac{1}{2}\omega_{\alpha\beta} \wedge [\omega_{\delta}^{\gamma} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta}) + \omega_{\delta}^{\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\gamma})],\\
\underset{g}{\star}\mathcal{S}^{\gamma} &= \frac{1}{2}\omega_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\gamma}),
\end{aligned} \tag{562}$$

where we use Eq.(523) for packing part of complicated formulas involving $\mathfrak{g}^{\alpha}$ and $d\mathfrak{g}^{\alpha}$ in the form of $\omega_{\alpha\beta}$. With Eqs.(562) we can provide a simple proof of Eq.(550). Indeed, let us compute $-2\underset{g}{\star}\mathcal{G}^{\delta}$ using Eq.(547) and Cartan's second structure equation.

$$\begin{aligned}
-2\underset{g}{\star}\mathcal{G}^{\delta} &= d\boldsymbol{\omega}_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta}) + \boldsymbol{\omega}_{\alpha\gamma} \wedge \boldsymbol{\omega}_{\beta}^{\gamma} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta})\\
&= d[\boldsymbol{\omega}_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta})] + \boldsymbol{\omega}_{\alpha\beta} \wedge d\underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta})\\
&+ \boldsymbol{\omega}_{\alpha\gamma} \wedge \boldsymbol{\omega}_{\beta}^{\gamma} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta})
\end{aligned}$$

[70]Those objects are called in [95], the Sparling forms [94]. However, they was already used much earlier by Thirring and Wallner in [97].

$$
\begin{aligned}
&= d[\boldsymbol{\omega}_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta})] - \boldsymbol{\omega}_{\alpha\beta} \wedge \boldsymbol{\omega}_{\rho}^{\alpha} \wedge \underset{g}{\star}(\mathfrak{g}^{\rho} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta}) \\
&- \boldsymbol{\omega}_{\alpha\beta} \wedge \boldsymbol{\omega}_{\rho}^{\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\rho} \wedge \mathfrak{g}^{\delta}) - \boldsymbol{\omega}_{\alpha\beta} \wedge \boldsymbol{\omega}_{\rho}^{\delta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\rho}) \\
&+ \boldsymbol{\omega}_{\alpha\gamma} \wedge \boldsymbol{\omega}_{\beta}^{\gamma} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta}) \\
&= d[\boldsymbol{\omega}_{\alpha\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\delta})] - \boldsymbol{\omega}_{\alpha\beta} \wedge [\boldsymbol{\omega}_{\rho}^{\delta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\beta} \wedge \mathfrak{g}^{\rho}) \\
&+ \boldsymbol{\omega}_{\rho}^{\beta} \wedge \underset{g}{\star}(\mathfrak{g}^{\alpha} \wedge \mathfrak{g}^{\rho} \wedge \mathfrak{g}^{\delta})] \\
&= 2(d \underset{g}{\star} \mathcal{S}^{\delta} + \underset{g}{\star} t^{\delta}).
\end{aligned} \tag{563}
$$

F A Comment on the Gauge Theory of Gravitation of References [21] and [48]

In Footnote 1 we recalled that in *GRT* a gravitational field is defined by an equivalence class of pentuples, where $(M, \boldsymbol{g}, D, \tau_{g}, \uparrow)$ and $(M', \boldsymbol{g}', D', \tau_{g}', \uparrow')$ are said equivalent if there is a diffeomorphism $f : M \to M'$, such that $\boldsymbol{g}' = f^{*}\boldsymbol{g}$, $D' = f^{*}D$, $\tau_{g}' = f^{*}\tau_{g}, \uparrow' = f^{*} \uparrow$, (where f^{*} here denotes the pullback mapping). For more details, see, e.g., [86, 79, 82]. Moreover, in *GRT* when one is studying the coupling of the matter fields, represented, say by some sections of the exterior algebra bundle[71] $\psi_{1,\dots,}\psi_{n}$, which satisfy certain coupled differential equations (*the equations of motion*[72]), two models $\{(M, \boldsymbol{g}, D, \tau_{g}, \uparrow), \psi_{1,\dots,}\psi_{n}\}$ and $(M', \boldsymbol{g}', D', \tau_{g}', \uparrow'), \psi'_{1,\dots,}\psi'_{n}\}$ are said to be dynamically equivalent iff there is a diffeomorphism $\mathtt{h} : M \to M'$, such that $(M, \boldsymbol{g}, D, \tau_{g}, \uparrow)$ and $(M', \boldsymbol{g}', D', \tau_{g}', \uparrow')$ and moreover $\psi'_{i} = f^{*}\psi_{i}$, $i = 1, \dots, n$.

This kind of equivalence is not a particularity of *GRT*, it is an obvious mathematical requirement that any theory formulated with tensor fields living in an arbitrary manifold must satisfy[73]. This fact is sometimes confused with the fact that for particular theory, say $\mathfrak{T}$ it may happen that there are diffeomorphisms $f : M \to M$ such that for a given model $\{(M, \boldsymbol{g}, D, \tau_{g}, \uparrow), \psi_{1,\dots,}\psi_{n}\}$ it is $\boldsymbol{g}' = f^{*}\boldsymbol{g}$, $D' = f^{*}D$, $\tau_{g}' = f^{*}\tau_{g}, \uparrow' = f^{*} \uparrow$, but $\psi'_{i} \neq f^{*}\psi_{i}$. When this happen

[71] If the exterior algebra bundle $\bigwedge T^{*}M$ is viewed as embedded in the Clifford algebra bundle $\mathcal{C}\ell(M, \mathtt{g})$ ($\bigwedge T^{*}M \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$) then even spinor fields can be represented in that formalism as some appropriate equivalence classes of sums of even non homegeneous differential forms. Details may be found in [82].

[72] The equations of motion, for any particular problem satisfy appropriate initial and boundary conditions.

[73] Some people now call this property background independence, although we do not think that to be a very appropriate name. Others call this property general covariance. Eventually it is most appropriate to simply say that the theory is invariant under diffeomorphisms. Moreover, we recall that there is an almost trivial mathematical theorem saying that the *action functional* for the matter fields represented by sections of the exterior algebra bundle plus the gravitational field is invariant under the action of one-parameter group of diffeomorphisms for Lagrangian densities that vanish on the boundary of the region where the action functional is defined. Details, e.g., in [82].

the set of diffeomorphisms satisfying that property defines a group, called a symmetry group of $\mathfrak{T}$, and of course, knowing the possible symmetry groups of a given theory facilitates the finding of solutions for the equations of motion [74].

Having saying that, we now analyze first the motivations for the 'gauge theory of the gravitational field' in Minkowski spacetime of references [21] and [48] formulated on a vector manifold U [75]. Those authors taking advantage of the obvious identification of the tangent spaces of U write that for a diffeomorphism $f : U \to U$, $x \mapsto f(x)$ we have some particularly useful representations for the pullback mapping (f^*, $f^*_{x'} : T^*_{x'}U \to T^*_x U$ and pushforward mapping (f_*, $f_{*x} : T_x U \to T_{x'}U$). Indeed, for representing the pushforward mapping [21] and [48] introduce the operator $\mathtt{f}$ such that

$$\begin{array}{rcccc} \mathtt{f} & : & \sec\bigwedge^1 TU & \to & \sec\bigwedge^1 TU, \\ & & a & \mapsto & \mathtt{f}(a) = a \underset{\eta}{\cdot} \partial_x f \end{array} \tag{564}$$

where

$$\partial_x = \gamma^\mu \frac{\partial}{\partial \mathtt{x}^\mu}, \tag{565}$$

with $\{\mathtt{x}^\mu\}$ coordinates for U in the Einstein-Lorentz-Poincaré gauge and $\{\gamma^\mu\}$ the dual basis of the basis $\{\vartheta_\mu\}$, as introduced in Chapter 5[76]. For representing the pullback mapping [21] and [48] introduce the operator $\mathtt{f}^\dagger$ such that[77]

$$\begin{gathered} \underline{\mathtt{f}}^\dagger : \sec\bigwedge T_x U \to \sec\bigwedge T_x U, \\ \psi_i(x) \mapsto \psi'_i(x) = (\underline{\mathtt{f}}^\dagger \psi_i)(x) = \psi_i(f(x)) \end{gathered} \tag{566}$$

Next those authors argue that the description of physics ψ_i or $\psi_i^{'}$ is not *covariant.* To cure this 'deficiency' they propose to introduce a gauge field $\boldsymbol{h} : \sec\bigwedge^1 TU \to \sec\bigwedge^1 TU$ such that,

$$\begin{aligned} \psi_i(x) &= \boldsymbol{h}^\dagger[\Psi_i(x)], \\ \psi'_i(x) &= \boldsymbol{h}^{\dagger\prime}[\Psi_i(x)], \end{aligned} \tag{567}$$

where $\Psi_i(x)$ is said to be the covariant description of $\psi_i(x)$. This implies in the following transformation law for $\boldsymbol{h}$ under a diffeomorphism $f : U \to U$,

$$\boldsymbol{h}^{\dagger\prime} = \underline{\mathtt{f}}^\dagger \boldsymbol{h}^\dagger. \tag{568}$$

From this point, using heuristic arguments, those authors say that this permits the introduction of a new metric extensor field on U, that they write as

$$\boldsymbol{g} = \boldsymbol{h}^\dagger \boldsymbol{h}, \tag{569}$$

[74]See some examples, e.g., in Chapter 5 of [82].

[75]The vector manifod U is as defined in Chapter 5, but with the elements of U being vectors instead of forms, and the elements of $\bigwedge TU$ being multivector fields instead of multiform fields.

[76]The reciprocal basis of $\{\gamma^\mu\}$ is the basis $\{\gamma_\mu\}$ such that $\gamma_\mu \underset{\eta}{\cdot} \gamma^\nu = \delta^\nu_\mu$.

[77]Recall that $\underline{\mathtt{f}}^\dagger$ is the extended of $\mathtt{f}$.

because differently from what we did in this paper they choose as basic algebra for performing calculations $\mathcal{C}\ell(U,\eta)$, instead of $\mathcal{C}\ell(U,\cdot)$. Take notice also that in [21, 48] it is $\boldsymbol{h} = h^{-1}$.

The reasoning behind Eqs.(567) is that the field $\Psi_i \in \sec \bigwedge TU$ has been selected as a representative of the class of the diffeomorphically equivalent fields $\{\underline{\mathtt{f}}^{\dagger}\psi_i\}$, for $f\ \in$DiffU, the diffeomorphism group of U. This is a very important point to take in mind. Indeed, in [21] and [48] in a region $V \subset U$ the gravitational potentials (called *displacement gauge invariant frame* $\{g^{\mu}\}$) are introduced as follows. First, take arbitrary coordinates $\{x^{\mu}\}$ covering V and introduce the coordinate vector fields

$$e_{\mu} := \frac{\partial x}{\partial x^{\mu}} = \partial_{\mu}x, \tag{570}$$

where the position vector $x := \mathtt{x}^{\mu}(x^{\alpha})\gamma_{\alpha}$ and the reciprocal basis of $\{e_{\mu}\}$ is the basis $\{e^{\mu}\}$ such that[78]

$$e^{\mu} := \partial_x x^{\mu}. \tag{571}$$

Then,

$$g_{\mu} := \boldsymbol{h}^{\dagger}(e_{\mu}) = h_{\mu}^{\alpha}e_{\alpha}, \tag{572}$$

$$g^{\mu} = \boldsymbol{h}^{-1}(e^{\mu}) = (\boldsymbol{h}^{-1}\partial_x)x^{\mu} \tag{573}$$

and

$$\boldsymbol{h}^{\dagger}(e_{\mu}) \underset{\eta}{\cdot} \boldsymbol{h}^{\dagger}(e_{\nu}) = \boldsymbol{g}(e_{\mu}) \underset{\eta}{\cdot} e_{\nu} = g_{\mu\nu}, \tag{574}$$

$$\boldsymbol{h}^{-1}(e^{\mu}) \underset{\eta}{\cdot} \boldsymbol{h}^{-1}(e^{\nu}) = \boldsymbol{g}^{-1}(e^{\mu}) \underset{\eta}{\cdot} e^{\nu} = g^{\mu\nu}. \tag{575}$$

Form its definition, the commutator $[e_{\mu}, e_{\nu}] := e_{\mu} \underset{\eta}{\cdot} \partial_x e_{\nu} - e_{\nu} \underset{\eta}{\cdot} \partial_x e_{\mu} = 0$, but in general we have that

$$\begin{aligned} [g_{\mu}, g_{\nu}] = [\boldsymbol{h}^{\dagger}e_{\mu}, \boldsymbol{h}^{\dagger}e_{\nu}] &:= (\boldsymbol{h}^{\dagger}e_{\mu} \underset{\eta}{\cdot} \partial_x)\boldsymbol{h}^{\dagger}e_{\nu} - (\boldsymbol{h}^{\dagger}e_{\nu} \underset{\eta}{\cdot} \partial_x)\boldsymbol{h}^{\dagger}e_{\mu} \\ &= (\partial_{\mu}h_{v}^{\alpha} - \partial_{\nu}h_{\mu}^{\alpha})e_{\alpha} = c_{\mu\nu}^{\alpha}e_{\alpha} \neq 0, \end{aligned} \tag{576}$$

i.e., $\{g_{\mu}\}$ is not a coordinate basis, because the structure coefficients $c_{\mu\nu}^{\alpha}$ of the basis g_{μ} are non null.

Remark E1 Before proceeding, we recall that at page 477 of [21] it is stated that the main property that must be required from a nontrivial field strength $\boldsymbol{h}$ is that we should not have $\boldsymbol{h}^{\dagger}(a) = \mathtt{f}^{\dagger}(a)$ where $\mathtt{f}$ is obtained from a diffeomorphism $f : U \to U$. This implies that $\partial_x \wedge \boldsymbol{h}^{\dagger}(e_{\mu}) \neq 0$, i.e., $(\partial_{\mu}h_{v}^{\alpha} - \partial_{\nu}h_{\mu}^{\alpha})e_{\alpha} \neq 0$.

´The next step in [21] towards the formulation of their gravitational theory is the introduction of what those authors call a 'gauge covariant derivative' that 'converts ∂_{μ} into a covariant derivative'. If $M \in \sec \bigwedge TU$ they define

[78] Recall that $\partial_x = \gamma^{\mu}\frac{\partial}{\partial \mathtt{x}^{\mu}} = e^{\mu}\frac{\partial}{\partial x^{\mu}}$, with $e^{\mu} = \gamma^{\alpha}\frac{\partial x^{\mu}}{\partial \mathtt{x}^{\alpha}}$.

$$\boxed{\mathbf{D}_\mu M = \partial_\mu + \Omega_\mu \times M,} \tag{577}$$

with $\Omega : \sec\bigwedge^1 TU \to \sec\bigwedge^2 TU$, $\Omega_\mu = \Omega(e_\mu)$. Before going on we must observe that the correct notation for $\mathbf{D}_\mu$ in order to avoid confusion[79] must be D_{e_μ} and we shall use it in what follows, thus to start, we write

$$\mathbf{D}_{e_\mu} M = \partial_\mu + \Omega_\mu \times M. \tag{578}$$

We also introduce the basis $\{\theta^\mu\}$ of T^*U dual of the basis $\{e_\mu\}$ of TU and also the basis $\{\zeta^\mu\}$ of T^*U dual of the basis $\{g_\mu\}$ of TU and write like in [21, 48]:

$$\mathbf{D}_{e_\mu} g_\nu := L^\alpha_{\mu\nu} g_\alpha, \quad \mathbf{D}_{e_\mu} g^\alpha = -L^\alpha_{\mu\nu} g^\nu, \quad \mathbf{D}_{e_\mu} \zeta^\alpha = -L^\alpha_{\mu\nu} \zeta^\nu \tag{579}$$

We also write

$$\mathbf{D}_{e_\mu} e_\nu := \mathbf{L}^\alpha_{\mu\nu} g_\alpha, \quad \mathbf{D}_{e_\mu} e^\alpha = -\mathbf{L}^\alpha_{\mu\nu} e^\nu, \quad \mathbf{D}_{e_\mu} \theta^\alpha = -\mathbf{L}^\alpha_{\mu\nu} \theta^\nu. \tag{580}$$

Note now that whereas the indices μ and ν in $\mathbf{L}^\alpha_{\mu\nu}$ refers to the same basis[80], namely $\{e_\mu\}$ the indices μ and ν in $L^\alpha_{\mu\nu}$ are hybrid[81], i.e., the first (μ) corresponds to the basis $\{e_\mu\}$ whereas the second (ν) corresponds to the basis $\{g_\nu\}$. This observation is very important, for indeed, if we write explicitly $\boldsymbol{\eta}$ and $\boldsymbol{g}$, the metric tensors which correspond to the metric extensors η^{-1} and $\boldsymbol{g}^{-1}$, we have immediately from $g_\mu \underset{\eta}{\cdot} g_\nu = g_{\mu\nu}$ and $e_\mu \underset{g}{\cdot} e_\nu = g_{\mu\nu}$ that

$$\boldsymbol{\eta} = g_{\mu\nu} \zeta^\mu \otimes \zeta^\nu, \quad \boldsymbol{g} = g_{\mu\nu} \theta^\mu \otimes \theta^\nu, \tag{581}$$

with $\{\zeta^\mu\}$ the dual basis of $\{g_\mu\}$ and $\{\theta^\mu\}$ the dual basis of $\{e_\mu\}$.

We then get

$$\begin{aligned} \mathbf{D}_{e_\mu} \boldsymbol{\eta} &= \mathbf{D}_{e_\mu}(g_{\alpha\beta} \zeta^\alpha \otimes \zeta^\beta) \\ &= (\partial_\mu g_{\alpha\beta} - g_{\rho\beta} L^\rho_{\mu\alpha} - g_{\alpha\rho} L^\rho_{\mu\beta}) \zeta^\alpha \otimes \zeta^\beta, \end{aligned} \tag{582}$$

i.e., the connection D will be metric compatible with $\boldsymbol{\eta}$ iff

$$\partial_\mu g_{\alpha\beta} - g_{\rho\beta} L^\rho_{\mu\alpha} - g_{\alpha\rho} L^\rho_{\mu\beta} = 0. \tag{583}$$

Now, in [21, 48] Eq.(583) is trivially true[82], but it is interpreted by those authors as meaning that $\mathbf{D}$ is compatible with $\boldsymbol{g}$. The error in those papers arises due

[79] In [48] Eq.(578) is written $D_\mu M = \partial_\mu + \omega(g_\mu) \times M$. We must immediately conclude that $\omega(g_\mu) = \omega(\boldsymbol{h}^\dagger(e_\mu)) = \Omega(e_\mu)$, i.e., $\omega = \boldsymbol{h}^\clubsuit \Omega$. Moreover from the notation used in [48] we should not infer that D_μ is representing D_{g_μ}, because if that was the case then the defining equation for D_{g_μ} should read $D_{g_\mu} M = h^\alpha_\mu \partial_\mu + \Omega(g_\mu) \times M$.

[80] Those indices are holonomic, since $\{e_\mu\}$ is a coordinate basis.

[81] The indice μ is holonomic and the indice ν is non holonomic.

[82] Indeed, we have for one side that $D_{e_\mu}(g_\alpha \underset{\eta}{\cdot} g_\beta) = \partial_\mu g_{\alpha\beta}$ and on the other side $D_{e_\mu}(g_\alpha \underset{\eta}{\cdot} g_\beta) = D_{e_\mu} g_\alpha \underset{\eta}{\cdot} g_\beta + g_\alpha \underset{\eta}{\cdot} D_{e_\mu} g_\beta = g_{\rho\beta} L^\rho_{\mu\alpha} + g_{\alpha\rho} L^\rho_{\mu\beta}$.

the confusion with the hybrid indices (holonomic and non holonomic) in $L^{\rho}_{\mu\alpha}$, and we already briefly mentioned in [27]. And, indeed, a trivial calculation shows immediately that

$$\mathbf{D}_{e_\mu}\boldsymbol{g}\neq 0, \tag{584}$$

i.e., the connection D has a non null nonmetricity tensor relative to $\boldsymbol{g}$. Indeed, we know from the results of Section 4.9.1 that the connection which is metric compatible with $\boldsymbol{g}$ is $\mathbf{D}'$, the $\boldsymbol{h}$-deformation of $\mathbf{D}$ such that if $a, b \in \sec \bigwedge^1 TM$,

$$\mathbf{D}'_a b = \boldsymbol{h}^{-1}(\mathbf{D}_a \boldsymbol{h}(b)). \tag{585}$$

Another serious equivoque in [21, 48] is the following. Those authors introduce a Dirac like operator, here written as $\boldsymbol{D}{=}g_\alpha \mathbf{D}_{e_\alpha}$. Then it is stated (see, e.g., Eq.(131) in [48]) that

$$\boldsymbol{D}\wedge g^\alpha = \frac{1}{2}(L^{\alpha}_{\mu\nu} - L^{\alpha}_{\nu\mu})g^\nu \wedge g^\mu \tag{586}$$

is the torsion tensor (more precisely the torsion 2-forms Θ^α). However, since the structure coefficients of the basis $\{g_\mu\}$ are non null the correct expression for the the torsion 2-form fields Θ^α are, as well known (see. e.g., [82])

$$\Theta^\alpha = \frac{1}{2}(L^{\alpha}_{\mu\nu} - L^{\alpha}_{\nu\mu} - c^{\alpha}_{\mu\nu})g^\nu \wedge g^\mu. \tag{587}$$

It follows that even assuming $L^{\alpha}_{\mu\nu} = L^{\alpha}_{\nu\mu}$ the connection $\mathbf{D}$ has torsion, contrary to what is claimed in [48]. Of course, a simple calculation shows that the Riemann curvature tensor of $\mathbf{D}$ is also non null.

The above comments clearly show that the gravitational theory presented in [21, 48] introduces a connection $\mathbf{D}$ that is never compatible with $\boldsymbol{g}$ and moreover does not reduce to *GRT* even when it happens that $\Theta^\alpha = 0$, and as such that theory seems to be *unfortunately* inconsistent[83].

G The Gravitational Field of a Point Mass Represented by the Nonmetricity Tensor of a Connection

In this section to furhter illustrate the point of view used in this paper, we give a model for gravitational field of a point mass, where that field is represented by the nonmetricity tensor of a special connection. To start, suppose that the world manifold M is a 4-dimensional manifold diffeomorphic to $\mathbb{R}^4$. Let moreover $(t, x, y, z) = (x^0, x^1, x^2, x^3)$ be global Cartesian coordinates for M.

[83] At http://www.mrao.cam.ac.uk/~clifford/publications/index.html there is a list of articles (published in very good journals) written by the authors of [21] and collaborators and based on the their gravitational theory may be found.

Next, introduce on M two metric fields:

$$\boldsymbol{\eta} = dt \otimes dt - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3, \tag{588}$$

and

$$\begin{aligned}\boldsymbol{g} &= \left(1 - \frac{2m}{r}\right) dt \otimes dt \\ &- \left\{\left(1 - \frac{2m}{r}\right)^{-1} - 1\right\} r^{-2}\left[(x^1)^2 dx^1 \otimes dx^1 + (x^2)^2 dx^2 \otimes dx^2 + (x^3)^2 dx^3 \otimes dx^3\right] \\ &- dx^1 \otimes dx^1 + \left(1 - \frac{2m}{r}\right)^{-1} (dx^2 \otimes dx^2 - dx^3 \otimes dx^3).\end{aligned} \tag{589}$$

In Eq. (589)

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}. \tag{590}$$

Now, introduce $(t, r, \vartheta, \varphi) = (x^0, x'^1, x'^2, x'^3)$ as the usual spherical coordinates for M. Recall that

$$x^1 = r \sin\vartheta \cos\varphi,\ x^2 = r \sin\vartheta \sin\varphi,\ x^3 = r\cos\vartheta \tag{591}$$

and the range of these coordinates in η are $r > 0$, $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$. For $\boldsymbol{g}$ the range of the r variable must be $(0, 2m) \cup (2m, \infty)$.

As can be easily verified, the metric $\boldsymbol{g}$ in spherical coordinates is:

$$\boldsymbol{g} = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr - r^2 (d\vartheta \otimes d\vartheta + \sin^2\vartheta d\varphi \otimes d\varphi), \tag{592}$$

which we immediately recognize as the *Schwarzschild metric* of *GR*. Of course, $\boldsymbol{\eta}$ is a Minkowski metric on M.

As next step we introduce two *distinct* connections, $\mathring{D}$ and D on M. We assume that $\mathring{D}$ is the Levi-Civita connection of $\boldsymbol{\eta}$ in M and D is the Levi-Civita connection of $\boldsymbol{g}$ in M. Then, by definition (see, e.g., [82] for more details) the ammetricities tensors of $\mathring{D}$ relative to $\boldsymbol{\eta}$ and of D relative to $\boldsymbol{g}$ are null, i.e.,

$$\begin{aligned}\mathring{D}\boldsymbol{\eta} &= 0, \\ D\boldsymbol{g} &= 0.\end{aligned} \tag{593}$$

However, the nonmetricity tensor $\mathfrak{A}^{\eta} \in \sec T_3^0 M$ of $\mathring{D}$ relative to $\boldsymbol{g}$ is non null, i.e.,

$$\mathring{D}\boldsymbol{g} = \mathfrak{A}^{\eta} \neq 0, \tag{594}$$

and also the nonmetricity tensor $\mathbf{A}^g \in \sec T_3^0 M$ of D relative to $\boldsymbol{\eta}$ is non null, i.e.,

$$D\eta = \mathfrak{A}^g \neq 0. \tag{595}$$

We now calculate the components of $\mathbf{A}^{\eta}$ in the coordinated bases $\{\boldsymbol{\partial}_\mu\}$ for TM and $\{dx^\nu\}$ for T^*M associated with the coordinates (x^0, x^1, x^2, x^3) of M. Since $\mathring{D}$ is the Levi-Civita connection of the Minkowski metric $\boldsymbol{\eta}$ we have that

$$\begin{aligned} \mathring{D}_{\boldsymbol{\partial}_\mu}\boldsymbol{\partial}_\nu &= L^{\rho}_{\mu\nu}\boldsymbol{\partial}_\rho = 0, \\ \mathring{D}_{\boldsymbol{\partial}_\mu}dx^\alpha &= -L^{\alpha}_{\mu\nu}dx^\nu = 0. \end{aligned} \tag{596}$$

i.e., the connection coefficients $L^{\rho}_{\mu\nu}$ of $\mathring{D}$ in this basis are *null.* Then, $\mathfrak{A}^{\eta} = Q_{\mu\alpha\beta}dx^\alpha \otimes dx^\beta \otimes dx^\mu$ is given by

$$\begin{aligned} \mathfrak{A}^{\eta} = \mathring{D}\boldsymbol{g} &= \mathring{D}_{\boldsymbol{\partial}_\mu}\left(g_{\alpha\beta}dx^\alpha \otimes dx^\beta\right) \otimes dx^\mu \\ &= (\frac{\partial g_{\alpha\beta}}{\partial x^\mu})dx^\alpha \otimes dx^\beta \otimes dx^\mu. \end{aligned} \tag{597}$$

To fix ideas, recall that for Q_{100} it is,

$$\begin{aligned} Q_{100} &= \frac{\partial}{\partial x^1}\left(1 - \frac{2m}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}\right) \\ &= -2m\frac{\partial}{\partial x^1}\left(\frac{1}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}\right) \\ &= \frac{2mx^1}{[(x^1)^2 + (x^2)^2 + (x^3)^2]^{\frac{3}{2}}} = \frac{2mx^1}{r^3}, \end{aligned} \tag{598}$$

which is *non null* for $x^1 \neq 0$. Note that also that $Q_{010} = Q_{001} = 0$.

Remark G1 Note that using coordinates (Riemann normal coordinates $\{\xi^\mu\}$ covering $V \subset U \subset M$) naturally adapted to a reference frame $\mathbf{Z} \in \sec TV$[84] in free fall ($D_{\mathbf{Z}}\mathbf{Z} = 0$, $d\alpha \wedge \alpha = 0$, $\alpha = g(\mathbf{Z},\ )$) it is possible, as well known, to put the connection coefficients of the Levi-Civita connection D of $\boldsymbol{g}$ equal to zero in all points of the world line of a free fall observer (an *observer* is here modelled as an integral line σ of a reference frame $\mathbf{Z}$, where $\mathbf{Z}$ is a time like vector field pointing to the future such that $\mathbf{Z}|_\sigma = \sigma_*$).

In the Riemann normal coordinates system covering $U \subset M$, not all the connection coefficients of the non metric connection $\mathring{D}$ (relative to g) are null. Also (of course) the nonmetricity tensor $\mathfrak{A}^{\eta}$ of $\mathring{D}$ (relative to $\boldsymbol{g}$) representing the true gravitational field is not null. An observer following σ does not fell any force along its world line because the gravitational force represented by the nonmetricity field is compensated by an inertial force represented by the non null connection coefficients[85] $L''^{\rho}_{\mu\nu}$ of $\mathring{D}$ in the basis $\{\frac{\partial}{\partial \xi^\nu}\}$.

The situation is somewhat analogous to what happens in any non inertial reference frame which, of course, may be conveniently used in *any* Special Relativity problem (as e.g., in a rotating disc [83]), where the connection coefficients of the Levi-Civita connection of $\boldsymbol{\eta}$ are not all null.

[84]For the mathematical definitions of reference frames, naturally adpted coordinates to a reference frame and observers, see. e.g., Chapter 5 of [82].

[85]$\nabla_{\frac{\partial}{\partial \xi^\mu}}\frac{\partial}{\partial \xi^\nu} = L''^{\rho}_{\mu\nu}\frac{\partial}{\partial \xi^\nu}$. The explicit form of the coefficients $L''^{\rho}_{\mu\nu}$ may be found in Chapter 5 of [82]

Standard clocks and standard rulers (standard according to *GRT*) are predict to have a behavior that seems to be (at least with the precision of the *GPS* system) exactly the observed one. However, but take into account that those objects are *not* the standard clocks and rulers of the model proposed here[86]. We would say that the gravitational field *distorts* the period of the standard clocks and the length of the standard rulers of *GRT* relative to the standard clocks and standard rules of the proposed model which has non null nonmetricity tensor[87]. But, who are the devices that materialize those concepts in the proposed model? Well, they are *paper concepts*, like the notion of time in some Newtonian theories. They are defined and calculated in order to make *correct* predictions. However, given the status of present technology we can easily imagine how to build devices for directly realizing the standard rulers and clocks of the proposed model.

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[86] Of course, our intention was only to illustrate our statement in the Intorduction that some gravitational fields may have many (equivalent) geometrical interpretations.

[87] On his respect see also [13].

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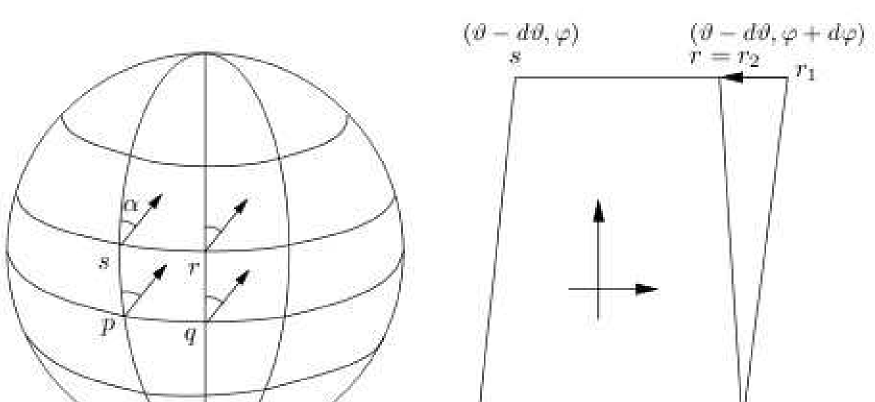
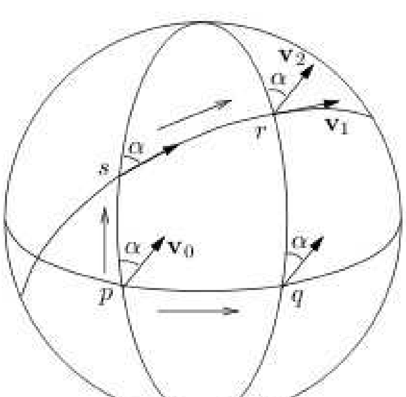
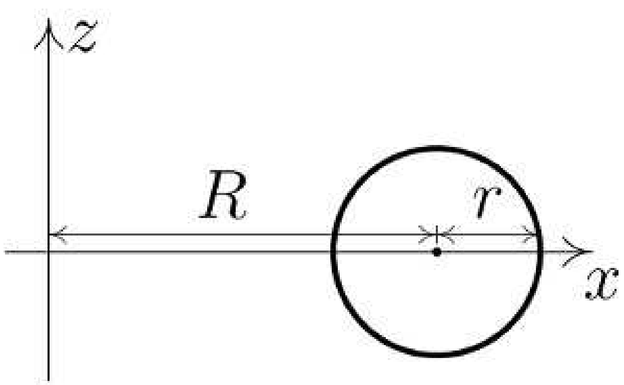


α
s
r
p
q
$(\vartheta - d\vartheta, \varphi)$
s
$(\vartheta - d\vartheta, \varphi + d\varphi)$
$r = r_2$
r_1
p
q
(ϑ, φ)
$(\vartheta, \varphi + d\varphi)$

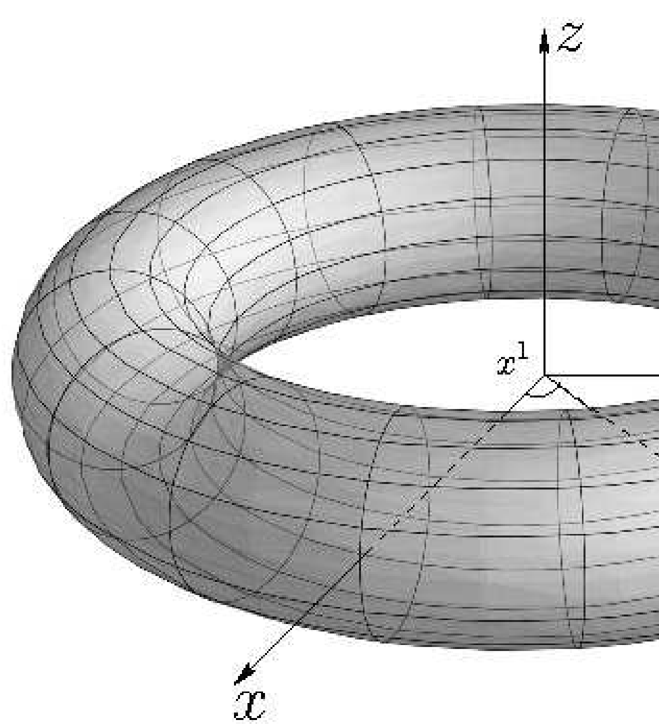
(a) (b)



$\mathbf{v}_2$
α
r
$\mathbf{v}_1$
s
α
α
$\mathbf{v}_0$
α
p
q



z
R
r
x



z
x^1
x